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DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

SIGNALS AND SYSTEMS
18EC45

STUDY MATERIAL
IV SEMESTER

Contents

1. MODULE I

Introduction and Classification of signals: Definition of signal and systems, communication and control system as examples Classification of signals.

Basic Operations on signals: Amplitude scaling, addition, multiplication, differentiation, integration, time scaling, time shift and time reversal.

Elementary signals/Functions: Exponential, sinusoidal, step, impulse and ramp functions. Expression of triangular, rectangularand other waveforms in terms of elementary signals.

2.MODULE II

System Classification and properties: Linear-nonlinear, Time variant-invariant, causal-noncausal, static-dynamic, stable- unstable, invertible.

Time domain representation of LTI System: Impulse response, convolution sum, convolution integral.

Computation of convolution sum and convolution integral using graphical method for unit step and unit step, unit step and exponential, exponential and exponential, unit step and rectangular, and rectangular and rectangular.

3.MODULE III

LTI system Properties in terms of impulse response: System interconnection, Memoryless, Causal, Stable, Invertible and Deconvolution, and step response.

Fourier Representation of Periodic Signals: CTFS properties and basic problems.

4.MODULE IV

Fourier Representation of aperiodic Signals: Introduction to Fourier Transform &
DTFT, Definition and basic problems.

Properties of Fourier Transform: Linearity, Time shift, Frequency shift, Scaling, Differentiation and Integration, Convolution and Modulation, Parseval's theorem and problems on properties of Fourier Transform..

5.MODULE V

The Z-Transforms: Z transform, properties of the region of convergence, properties of the Z-transform, Inverse Z-transform, Causality and stability, Transform analysis of LTI systems.

UNIT 1 : INTRODUCTION

Definitions of a Signal & a System

- A Signal is defined as any physical quantity that varies with time, space or any other independent variables.

- A Signal is a function of one or more variables which conveys information on the nature of physical phenomenon (S&H)

- when the function depends on a single variable the signal is said to be one-dimensi

  Ex: A Speech Signal [whose Amplitude varies with time]

- when the function depends on 2 or more variables, the signal is said to be multi dimensional.

  Ex: An Image Signal (2D signal with Horizontal & Vertical co-ordinates).

More Examples for Signals

Signals, in one or another form constitutes basic ingredients of our daily lives.

(i) The common form of human communication takes place through the use of speech signals.

(ii) Another form is visual in nature, with signals taking the form of images of people or objects around us.

(iii) Another form is through electronic mail over internet.

(iv) ECG - Heart beat of a patient and monitoring his/her blood pressure and temperature, a doctor is able to diagnose the presence or absence of an illness or disease.

(v) By listening weather forecast over the radio, we hear references made to daily variations in temperature, humidity and speed and direction of the winds.

(vi)

System :

- A System is formally defined as an entity that manipulates one or more signals to accomplish a function, thereby yeilding new signals. (SH)

(OR)

- System is defined as a physical device that performs the operation on a signal.

Block diagram representation of a S/m.

I/p signal → System → O/p signal

- Here the i/p & o/p signals depends on the intended application of the S/m.

Ex: (1) Filter : it is used to reduce the noise and interference corrupting a desired information bearing signal. In this case filter performs some operation on the signal.

(2) In a communication S/m, the i/p signal may be a speech signal or computer data, the S/m is made of combination of Transmitter, channel & Receiver. The o/p

signal is an estimate of an Original message signal.

(3) In an Automatic speaker recognition S/m the i/p signal is a speech [voice] signal. The S/m is a Computer & the o/p signal is the identity of the speaker.

(4) In an aircraft landing S/m, the ip signal is the desired position of the aircraft relative to the runway, the S/m is the aircraft and the op signal is the correction to the lateral position of the aircraft.

* Single Valued means that for every instant of time there is a unique value of the function. This value may be a real number, we speak of real valued signal, or complex number, in which case we speak of Complex valued signal. In either case, the independent Variable, time is real valued.

CLASSIFICATION OF SIGNALS.

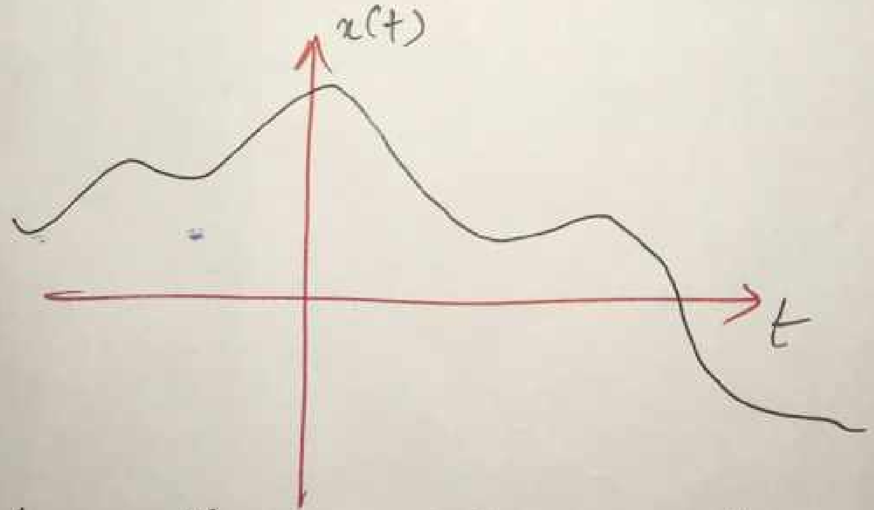
Note: Restrict to only one dimensional signals which are defined as single valued function of time.*

1. Continuous - Time Signals & Discrete - Time Signals.

[* How they are defined as a function]

→ A signal $x(t)$ is said to be continuous time signal, if it is defined for all time 't', i.e, whose amplitude or value varies continuously with time.

fig a : represents an example of a continuous Time signal.

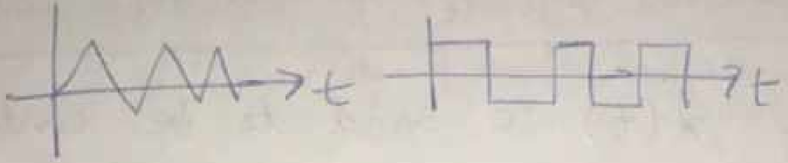


→ A continuous Time signals arises naturally when a physical w/f such as acoustic wave & light wave is converted into an electrical signal.

Representation of CT Signals.

1. Formula Method $x(t) = e^{-t}$
   $x(t) = \sin t$

2. Graphichal Method



- A discrete-time signal is defined only at discrete instants of time. In this case, the independent variable has discrete values only which are usually uniformly spaced.

- A discrete-time signal is often derived from continuous Time signal by sampling it at uniform rate.

- Let 'T' denotes the sampling period & 'n' denotes an integer (may be +ve or -ve)

- Sampling a CT signal $x(t)$ at time yields a sample of value $x(nT)$ i.e, a DT signal shown in fig (b)

We can write $x[n] = x(nT)$

$n = 0, \pm 1, \pm 2 \ldots$

- Thus DT signal is represented by the sequence no $[x[-2], x[-1], x[0], \ldots]$

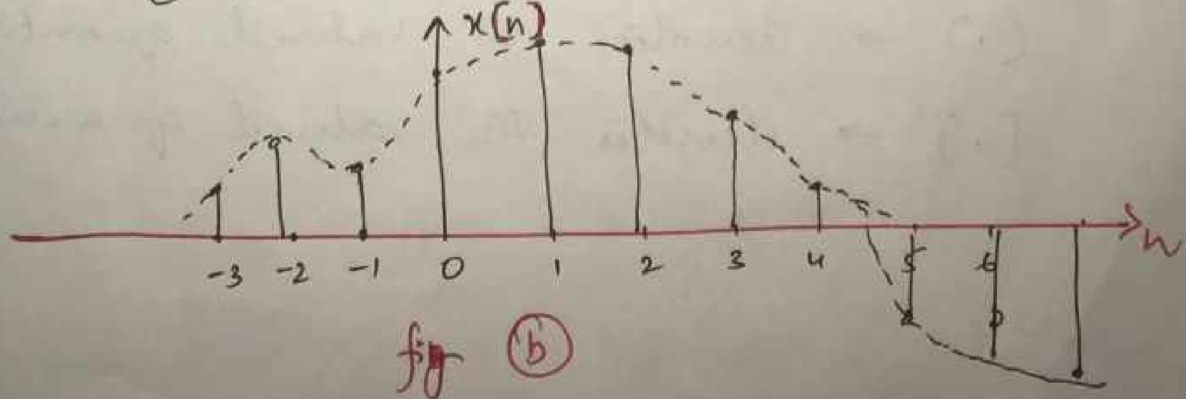


fig (b)

Representation of DT - signals.

1. Formula Method $x(n) = n^2$
$= n+1$

2. Graphical Method

3. Sequence method

$\{ \cdots 1, 0, \underset{\uparrow}{1}, -2 \; \cdots \; \cdots \}$

Note: $t \rightarrow$ time for CT - signal.
$n \rightarrow$ time for DT signal.
$(\cdot) \rightarrow$ denotes CT valued quantity
$[\cdot] \rightarrow$ denotes DT valued quantity.

(2) ANALOG & DIGITAL SIGNALS.

- If a CT signal $x(t)$ take on value in the continuous interval (a,b) where a may be $-\infty$ and b may be $+\infty$, then $x(t)$ is called as an Analog signal.

- If a DT signal $x[n]$ can take on only a finite no of distinct values then we call this signal as a digital signal.

(3) Even & ODD Signals

- A signal $x(t)$ or $x[n]$ is referred to as even signal

  if $x(t) = x(-t) \quad \forall t$

  $x[n] = x[-n] \quad \forall n$

where $x(t)$ - CT signal
$x[n]$ - DT signal.

- A signal $x(t)$ or $x[n]$ is referred to as ODD signal if

  $x(-t) = -x(t) \quad \forall t$

  $x[-n] = -x[n] \quad \forall n$

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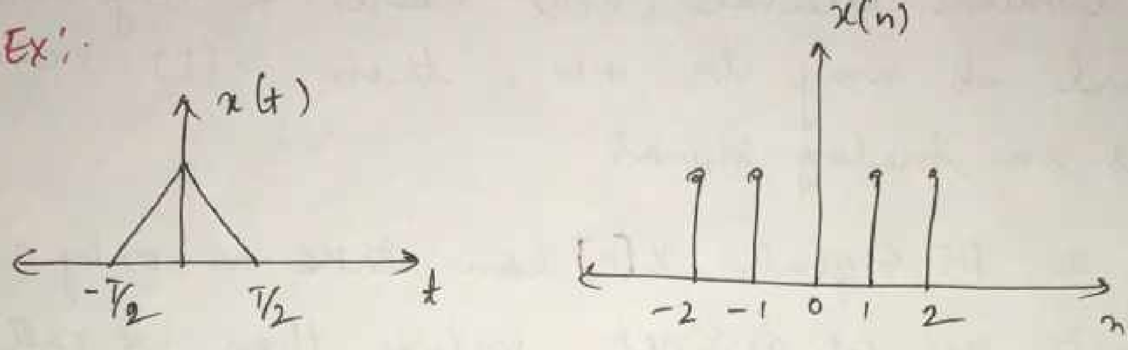
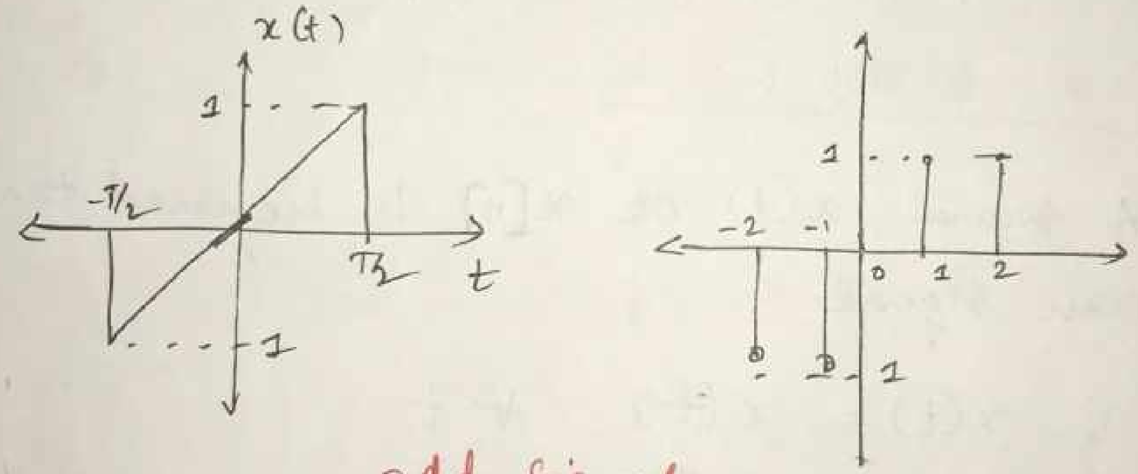
* Even signals are symmetric about the vertical axis or time origin, whereas odd signals are asymmetric about the origin.

Ex:



Even signals.



odd signals

Develop the Even/odd decomposition of a general signal $x(t)$ by applying definitions.

Let us consider an arbitrary CT signal $x(t)$.

Let $x(t) = x_e(t) + x_o(t)$ $\longrightarrow$ (1)

where $x_e(t) \rightarrow$ even part of $x(t)$

$x_e(-t) = x_e(t) \longrightarrow$ (2)

$x_o(t) \rightarrow$ odd part of $x(t)$

$x_o(-t) = -x_o(t) \longrightarrow$ (3)

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· Changing '$\underline{t}$' to '$\underline{-t}$' in eqn 1

we get, $x(-t) = x_e(-t) + x_o(-t) \rightarrow 4$

Substituting 2 & 3 in eqn 4

$$x(-t) = x_e(t) - x_o(t) \rightarrow 5$$

Adding eqns 1 & 5

$$x(t) + x(-t) = x_e(t) + x_e(t) + \underline{\cancel{x_o(t)} - \cancel{x_o(t)}}$$

$$= 2\,x_e(t)$$

or $\boxed{x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]} \rightarrow 6$

Subtracting eqn 5 from eqn 1 we get,

$$x(t) - x(-t) = \cancel{x_e(t)} + x_o(t) - \cancel{x_e(t)} + x_o(t)$$

$$x(t) - x(-t) = 2\,x_o(t)$$

or $\boxed{x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]} \rightarrow 7$

|||ly for DT signals

$$\boxed{\begin{aligned} x_e[n] &= \tfrac{1}{2}\left[x[n] + x[-n]\right] \rightarrow 8 \\ x_o[n] &= \tfrac{1}{2}\left[x[n] - x[-n]\right] \rightarrow 9 \end{aligned}}$$

(1) ST the product of 2 even signals or 2 odd signals is an even signals. while the product of an even and odd signal is an odd signal.

Solⁿ: Let $y[n] = y_1[n]\, y_2[n]$

(i) If $y_1[n]$ & $y_2[n]$ are both even then

$$y[-n] = y_1[-n]\, y_2[-n]$$
$$= y_1[n]\, y_2[n] = y[n]$$

$\therefore$ Thus $y(n)$ is even

(ii) If $y_1[n]$ & $y_2[n]$ are both odd, then

$$y[-n] = y_1[-n]\, y_2[-n]$$
$$= -y_1[n]\,(-y_2[n])$$
$$= y_1[n]\, y_2[n] = y[n]$$

(iii) If $y_1(n)$ - even & $y_2(n)$ - odd

$$y_3[-n] = y_1[-n]\, y_2[-n]$$
$$= y_1[n]\,[-y_2[n]]$$
$$= -y_1[n]\, y_2[n]$$
$$= -y[n].$$

$\therefore$ $y[n]$ is odd.

(2) Find the even & odd parts of $x(t) = e^{jt}$

Sol^n Let $x_e(t)$ & $x_o(t)$ be even & odd part of $x(t)$

Then $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$= \frac{1}{2}[e^{jt} + e^{-jt}]$

$= \underline{\cos t}$.

$\left\{ \cos t = \frac{e^{jt} + e^{-jt}}{2} \qquad \sin t = \frac{e^{jt} - e^{-jt}}{2j} \right\}$

$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$= \frac{1}{2}[e^{jt} - e^{-jt}] = \underline{j \sin t}$.

(4) Deterministic & Random Signals.

- Any signal that can be uniquely described by an explicit mathematical expressions, a table of data or a well defined rule is called deterministic.
- Thus a deterministic signal can be defined as "A signal about which there is no uncertainty with respect to its value @ any time" i.e, we can predict the value of the signal before its actual occurrence.

Ex: $x(t) = t + 3$, $x(t) = e^{-t}$, $x(n) = 2n+3$, $x(t) = \sin t$, $x(n) = 2^n$.

Random signals takes random values at any given time instant. They are unable to predict the values of the signal before its actual occurence.

Ex:- Noise, Speech signal, Audio signal.

(5) Periodic & Non-Periodic Signals.

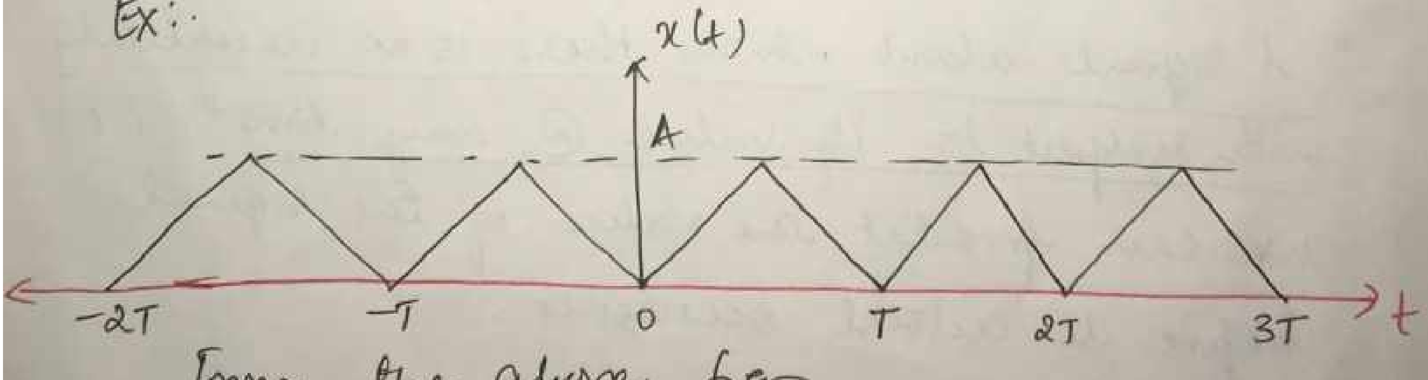
- A CT signal $x(t)$ is said to be periodic signal if it satisfies the condition

$$\boxed{x(t) = x(t+T) \quad \forall t} \rightarrow (1)$$

where T = +ve constant.

- Any signal whose amplitude value repeats after the certain amount of time is called a periodic signal

Ex:-



From the above fig

$$x(t+mT) = x(t) \quad \forall m$$

m = any integer

- The smallest value of T that satisfies eqn (1) is called the fundamental period of $x(t)$.

- Fundamental period defines the duration of one complete cycle of $x(t)$. Reciprocal of the fundamental period is called fundamental frequency.

- It describes how frequently the periodic signal repeats itself.

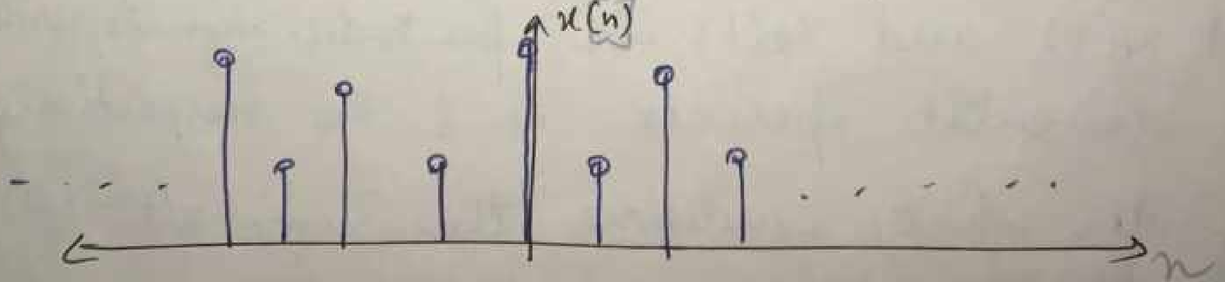
$$f = \frac{1}{T} \text{ Hz or cycles/sec.}$$

- Angular frequency is measured in rad/sec and is defined by $\omega = \frac{2\pi}{T} = 2\pi f$

Periodic DT Signal.

$$\boxed{x[n] = x(n+N) \quad \forall n} \rightarrow (2)$$

where N = +ve integer.



Angular frequency $\Rightarrow \Omega = \frac{2\pi}{N}$ rad/sec.

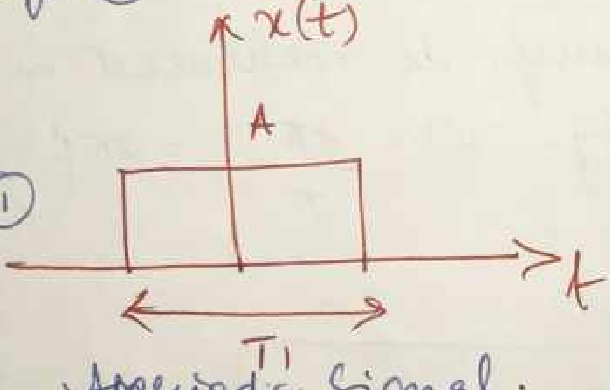
we can write

$$x[n+mN] = x[n] \quad \forall n$$

m = any integer.

The fundamental period of $x[n]$ is the smallest integer N for which eqn ② holds.

* Any sequence which is not periodic is called a non-periodic [aperiodic] sequence which is shown fig ①. i.e, there is no value of T to satisfy the condition in eqn ①



A = amplitude

T_1 = duration of signal.

fig ①

Aperiodic signal.

Periodicity of Sum of 2 Signals.

Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 & T_2 respectively.

(i) Under what conditions the Sum $x(t) = x_1(t) + x_2(t)$ is periodic.?

(ii) what is the fundamental period of $x(t)$, if it is periodic.

Soln :

Since $x_1(t)$ and $x_2(t)$ are periodic with fundamental periods T_1 & T_2, we have

$$x_1(t) = x_1(t+T_1) = x_1(t+mT_1)$$

$m \Rightarrow$ a positive integer.

$$x_2(t) = x_2(t+T_2) = x_2(t+nT_2)$$

$n \Rightarrow$ a positive integer.

Thus its Sum

$$x(t) = x_1(t) + x_2(t) \rightarrow (1)$$

$$= x_1(t+mT_1) + x_2(t+nT_2) \rightarrow (2)$$

In order for $x(t)$ to be periodic with period T, it is required that

$$\boxed{x(t+T) = x_1(t+T) + x_2(t+T)} \rightarrow (3)$$

$$= x_1(t+mT_1) + x_2(t+nT_2) \rightarrow (3)$$

Comparing (3) & (2)

Thus we must have

$$mT_1 = nT_2 = T \longrightarrow (4)$$

$$\frac{T_1}{T_2} = \frac{n}{m} = \text{a rational no} \rightarrow (5)$$

$\therefore$ The Sum of the 2 periodic Signals is periodic, only if the ratio of their respective fundamental periods can be expressed as a rational no.

(ii) Fundamental period is the LCM of T_1 & T_2.

(6) Energy and power signals.

- In an electrical S/m, a signal may be represented, as voltage or a current.
- Let us consider a voltage $v(t)$ developed across a resistor R, producing a current $i(t)$.

$i(t)$ R

$V(t)$

$$V(t) = i(t) \cdot R$$

if $R = 1\Omega$

$$V(t) = i(t) = x(t)$$

- The instantaneous power dissipated in the resistor is defined by

$$P(t) = \frac{V^2(t)}{R} = i^2(t) \cdot R$$

$$\therefore \boxed{P(t) = V^2(t) = i^2(t) = x^2(t)}$$

- The power dissipated in a 1Ω resistor is called as Normalized power.

- The total energy $\underline{\underline{E}}$ and the average power $\underline{\underline{P}}$ of the signal $x(t)$ are

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)\, dt \quad (1)$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \to (2)$$

If $x(t)$ is complex then eqn (1) & (2) are modified as

$$\cancel{E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \rightarrow (3)}$$

$$\cancel{\& \; P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \rightarrow (4)}$$

Not reqd.

If a signal $x(t)$ is periodic then eqn (4) gets modified as (fundamental period T)

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt \longrightarrow (5)$$

III If for a discrete time signal $x(n)$

$$E = \sum_{n=-\infty}^{\infty} x^2[n] \longrightarrow (6)$$

$$P = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N} x^2[n] \longrightarrow (7)$$

If $x(n)$ is periodic (fundamental period)

$$P = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

Conclusions

(1) A signal $x(t)$ or $x(n)$ is said to be a power signal, If and only if the avg power satisfies the following condn:

$$\boxed{0 < P < \infty \;:\; E = \infty}$$

(2) A signal $x(t)$ or $x(n)$ is said to be energy signal, if and only iff the total energy of the signal is a finite quantity

i.e, $\boxed{0 < E < \infty \quad , \quad P = 0}$

(3) Usually periodic signals and random signal are power signals.

(4) Signals that are both deterministic and non-periodic are energy signals.

BASIC OPERATIONS ON SIGNALS.

(1) Operations performed on dependent Variables.

Note: (1) Dependent Variables → corresponds to amplitude or value of the signal. [Y-axis]

(2) Independent Variables → is <u>time</u> or '<u>n</u>'

[X-axis] t for CTS
n for DTS.

(a) Amplitude Scaling.

Let $x(t)$ be a CTS. The signal $y(t)$ resulting from amplitude scaling is given by $\boxed{y(t) = c \cdot x(t)}$

where $c \to$ Scaling factor.

|||ly for DTS $\boxed{y[n] = c\, x[n]}$

Ex: Physical example of a device
- Amplifier – performs amplitude scaling operation
- Attenuator

Let $x(t) = 2t+3$

If $c = 2$

$$y(t) = c\, x(t) = 2[2t+3]$$
$$= 4t+6.$$

- Let $x[n] = [3, 4, \underset{\uparrow}{2}, 0, -2, 8]$

if $a = 2$

$$y[n] = a\,x[n] = 2[3, 4, \underset{\uparrow}{2}, 0, -2, 8]$$

$$= [6, 8, \underset{\uparrow}{4}, 0, -4, 16]$$

(b) Addition

Let $x_1(t)$ and $x_2(t)$ denote a pair of CTS, The signal $y(t)$ obtained by addition of $x_1(t)$ and $x_2(t)$ is defined by

$$\boxed{y(t) = x_1(t) + x_2(t)}$$

* at each and every instant of time.

IIIly for DTS

$$\boxed{y[n] = x_1[n] + x_2[n]}$$

Ex:: Audio mixer or Frequency Mixer — which combines low frequency and high frequency signals.. [Music + voice]

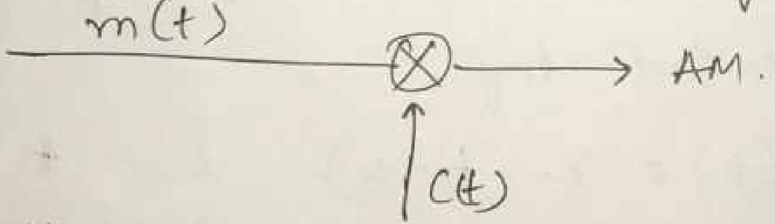
(3) Multiplication

Let $x_1(t)$ and $x_2(t)$ denote a pair of CTS. The signal $y(t)$ resulting from the multiplication of $x_1(t)$ and $x_2(t)$ is given by

$$y(t) = x_1(t) \cdot x_2(t).$$

III^ly for DTS $y[n] = x_1[n] \cdot x_2[n]$.

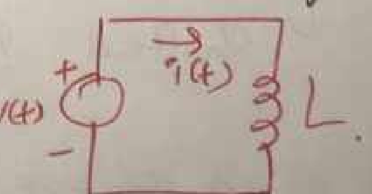
Ex: Multiplication operation is often performed in Analog communication i.e, modulation where an audio frequency signal is xed by a high frequency sinusoidal wave known as carrier. The resulting signal is AM wave



(4) Differentiation

Let $x(t) \rightarrow$ CTS, The derivative of $x(t)$ w.r.t time is defined by

$$y(t) = \frac{d\,x(t)}{dt}.$$



Ex: A physical device which performs differentiation is an inductor.

Voltage across inductor $v(t) = L \frac{di(t)}{dt}$

(5) Integration.

Let $x(t) \rightarrow$ CTS, the integral of $x(t)$ w.r.t time is defined by $y(t) = \int_{-\infty}^{t} x(t)\,dt$

This type of operation is performed in a capacitor. The voltage across the capacitor is given by

$$V(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau.$$

(2) Operations performed on independent variable (time)

(1) Time Shifting.

Let $x(t)$ be a CTS, its shifted version is represented by

$$y(t) = x(t-a) = x(t-t_0)$$

where $a \rightarrow$ amount of time shift.

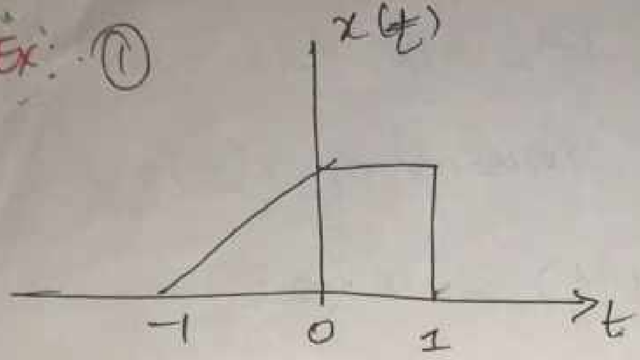
If a is +ve $(a>0)$ then the signal is shifted right by $\underline{a}$ units of time.

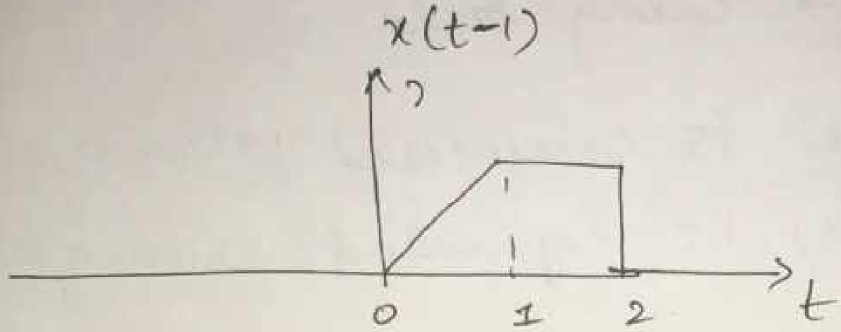
If a is -ve $(a<0)$, then the signal is shifted left by $\underline{\underline{a}}$ units of time.

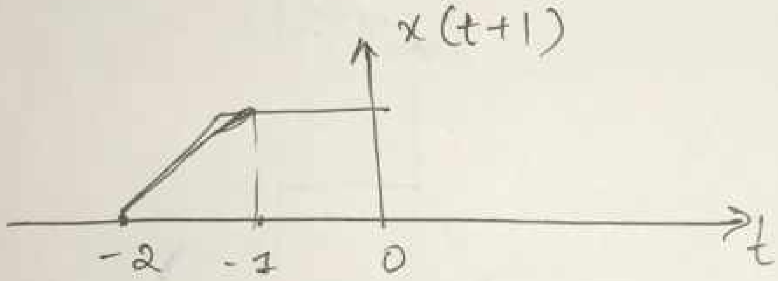
$|||^{ly}$ for DTS

$$y[n] = x[n-k]$$

Ex: 1







2. $x[n] = \{3, 2, 0, \underset{\uparrow}{4}, 6, 4\}$

$x[n-1] = \{3, 2, \underset{\uparrow}{0}, 4, 6, 4\}$

$x[n+1] = \{3, 2, 0, 4, \underset{\uparrow}{6}, 4\}$

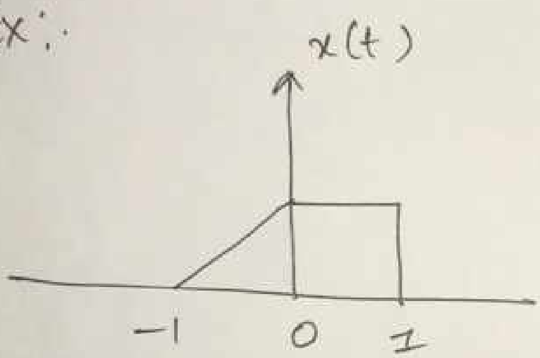
(2) Time Scaling : For a CTS $x(t)$, the time scaled version of $x(t)$ is represented by $y(t) = x(at)$

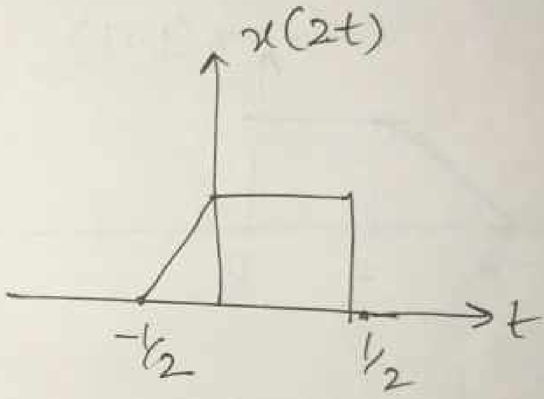
where $a \rightarrow$ Scaling factor.

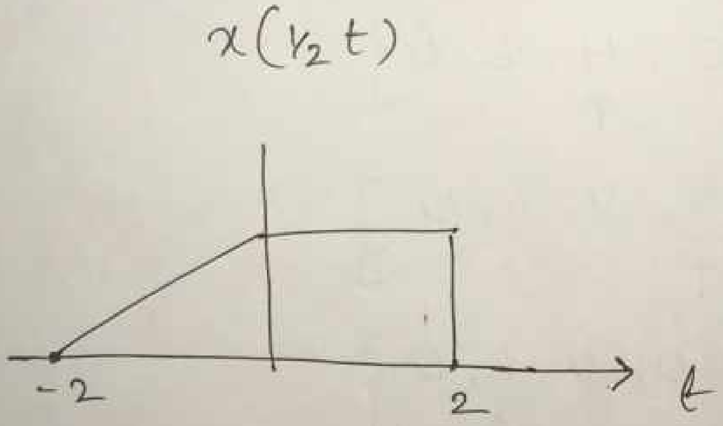
If $a > 1$, $y(t)$ is compressed version of $x(t)$

$a < 1$, $y(t)$ is expanded verision of $x(t)$

Ex:







IIIly for DTS

$$y[n] = x[an] \quad ; \quad a > 1$$

- defined only for integer values of n.
- Down Sampling.
- Due to down sampling some of the samples

of $x[n]$ are lost.

Ex:

(1) $x[n] = \{ \overset{-4}{2}, \overset{-3}{3}, \overset{-2}{4}, \overset{-1}{-3}, \underset{\uparrow}{\overset{0}{5}}, \overset{1}{1}, \overset{2}{6}, \overset{3}{7}, \overset{4}{-4}, \overset{5}{2}, \overset{6}{8} \}$

$y[n] = x[2n] \Rightarrow$ { represent a time scaled signal by a factor of 2 }

$y[n] = \{ 2, 4, \underset{\uparrow}{5}, 6, -4, 8 \}$

In this case even numbered samples are retained and odd numbered samples are lost.

(2) $g[n] = x[3n] = \{ 3, \underset{\uparrow}{5}, 7, 8 \}$

$g[-1] = x[-3]$
$g[0] = x[0]$
$g[1] = x[3]$
$g[2] = x[6]$

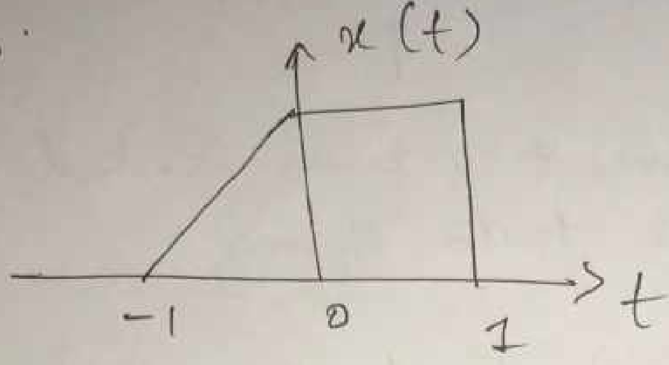
(3) Time Reversal / Time Folding.

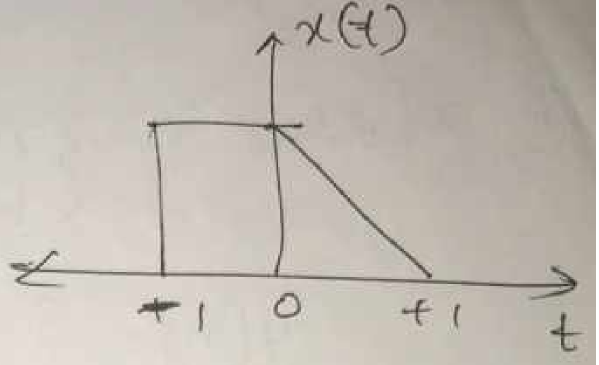
- The folded signal of any signal $x(t)$ is obtained by folding signal about the vertical axis at $t=0$.
- It is denoted by $x(-t)$

$$\boxed{y(t) = x(-t)}$$

Illy for DTS $\boxed{y[n] = x[-n]}$

Ex:

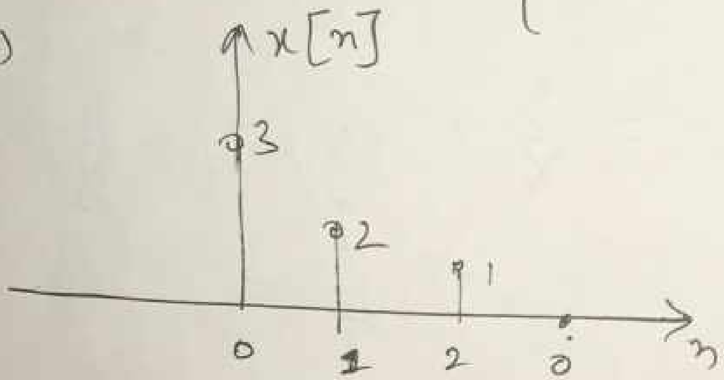


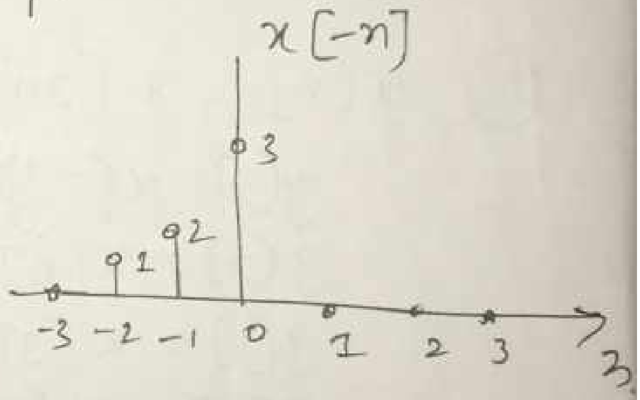


(1) $x[n] = \{3, 0, -3, \underset{\uparrow}{2}, 1, 6, 7\}$

$x[-n] = \{7, 6, 1, \underset{\uparrow}{2}, -3, 0, 3\}$

(2)





ELEMENTARY SIGNALS. **

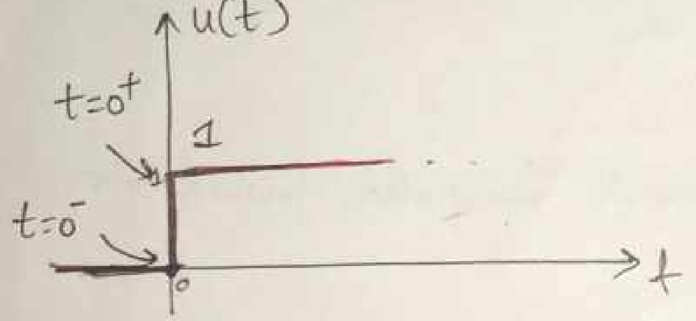
- serves as building blocks for the construction of more complex signals.
- used to model many physical signals that occur in nature.

BASIC CONTINUOUS TIME SIGNALS.

1. UNIT STEP FUNCTION.

The unit step function $u(t)$ is defined as

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

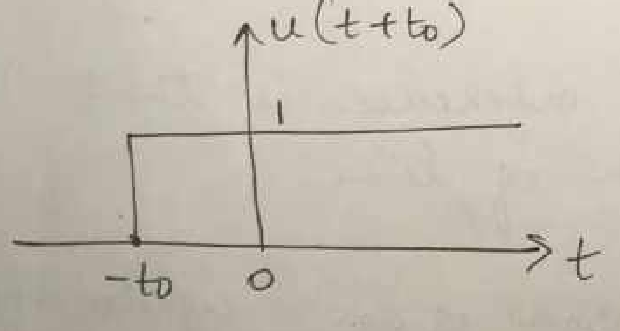


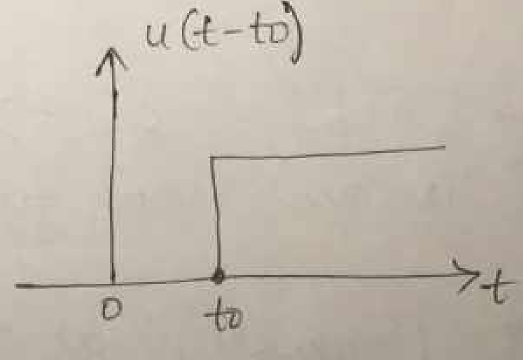
@ $t=0$, the function is discontinuous, since its value changes from 0 to 1

$$u(t) = \begin{cases} 1 & t \geq 0^+ \\ 0 & t \leq 0^- \end{cases}$$

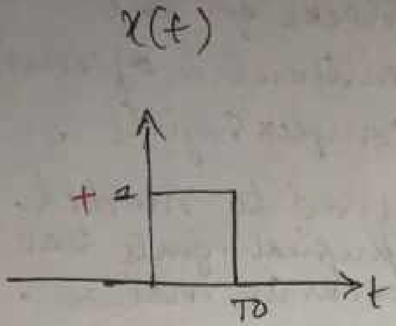
III^{ly} shifted unit step function is defined as follows.

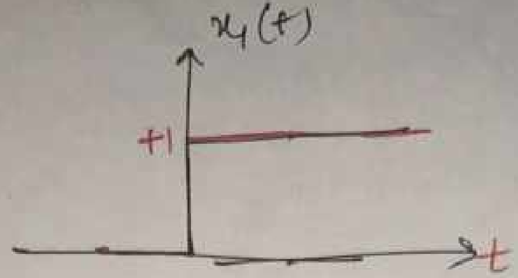
$$u(t+t_0) = \begin{cases} 1 & t+t_0 > 0 \\ 0 & t+t_0 < 0 \end{cases}$$

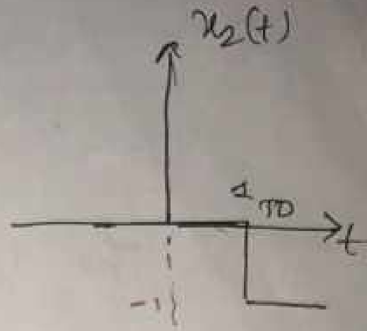




* The step function is used as a building block for several discontinuous w/f.







$$x(t) = x_1(t) + x_2(t)$$

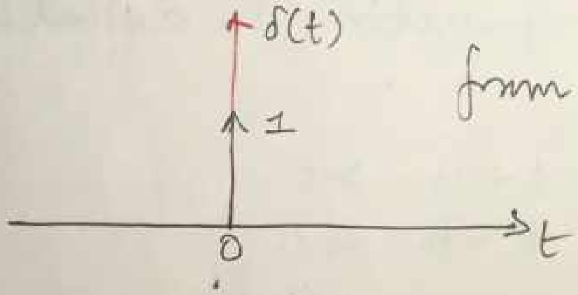
$$\boxed{x(t) = u(t) - u(t-T_0)}$$

(2) UNIT IMPULSE FUNCTION

* The continuous time unit impulse function is defined as follows.

$$\delta(t) = 0 \qquad t \neq 0 \longrightarrow (1)$$

and $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \longrightarrow (2)$$



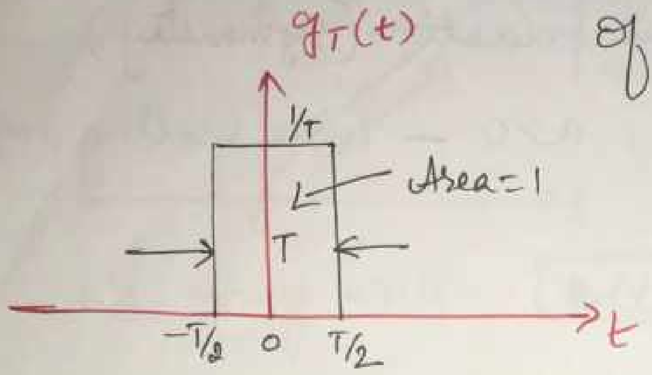
from fig $\Rightarrow \delta(t) = \begin{cases} 0, & t \neq 0 \\ 1 & t = 0. \end{cases}$

* $\delta(t) = \delta(-t)$ – inference is that $\delta(t)$ is an even function of time.

* Normally at the arrow of an impulse function we write a no. This no. corresponds to the area under the pulse known as the weight of the impulse and not the amplitude.

* The amplitude of the impulse function at $t=0$ is unbounded.

* We can also write an impulse as shown below. It has a narrow rectangular pulse of unit area..



* The width of the rectangular pulse is very small value $(T \to 0)$.

* Height is very large ($1/T$) value.

* ∴ The unit impulse is regarded as a rectangular pulse with a width that has become infinitely small and height that has become infinitely large maintaining the overall area at unity.

i.e, $\boxed{\delta(t) = \lim_{T \to 0} g_T(t)}$

— From eqn (1), $\delta(t)$ is zero everywhere except @ the origin.

* (2) ⇒ Area under the unit impulse is unity.

* $\delta(t)$ is also referred to as the Dirac delta function.

* $\delta(t)$ is the derivative of the step function $u(t)$ w.r.t time. $\frac{du(t)}{dt} = \delta(t)$

* Conversely, the step function $u(t)$ is the integral of $\delta(t)$ w.r.t time. $u(t) = \int \delta(t)\,dt$

Properties:

1. $\delta(t) = \delta(-t)$ - Even function (symmetry)

2. $\delta(at) = \frac{1}{a}\delta(t)$, $a>0$ – Time scaling property

$\delta(at) = \lim_{T\to 0} g_T(at)$

3. $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$ – shifting property

4. $\delta(t) = \frac{du(t)}{dt}$

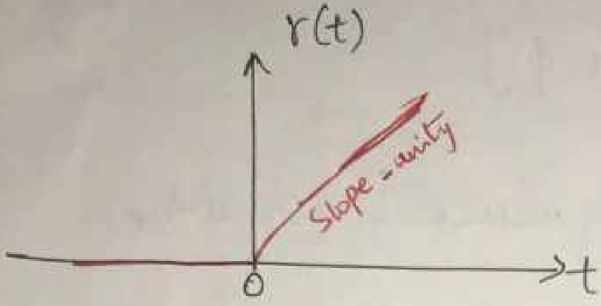
$\delta(t-t_0) = \frac{d[u(t-t_0)]}{dt}$

5. $x(t)\cdot\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

$x(t)$ - Continuous @ $t = t_0$.

(3). RAMP FUNCTION

* The integral of step function $u(t)$ is a ramp function of unit slope.



$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$r(t) = \int u(t)\,dt$$

We may write $\boxed{r(t) = t \cdot u(t)}$

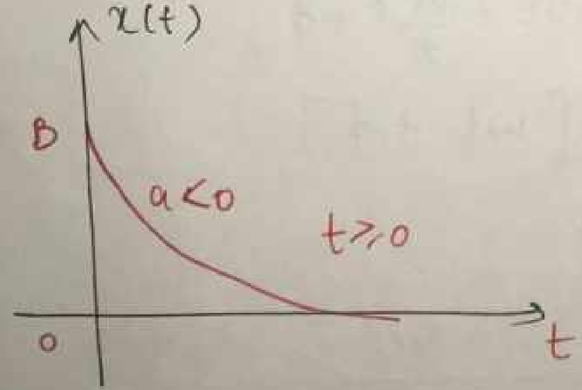
(4) EXPONENTIAL SIGNALS.

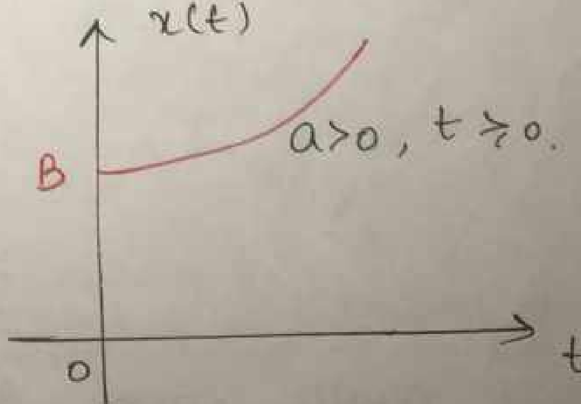
* A real exponential signal in its general form is written as

$x(t) = Be^{at}$. where B & a real parameters

B → Amplitude of signal $x(t)$ @ $t=0$.

* If $a<0$, $x(t)$ is said to be decaying exponential
$a>0$, $x(t)$ is said to be growing exponential.





(5) SINOSOIDAL SIGNALS.

General expression for continuous time Sinosoidal is

$$x(t) = A \cos[\omega t + \phi]$$

$A \rightarrow$ Amplitude

$\omega \rightarrow$ Angular frequency in rad/sec.

$\phi \rightarrow$ Phase angle in radians.

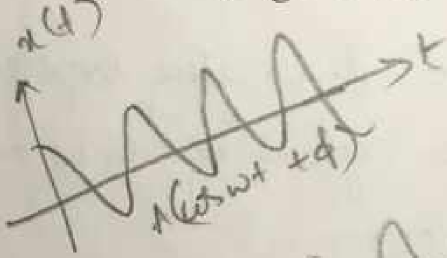
* The period of Sinosoidal signal is defined as

$$\omega = 2\pi f$$

$$T = \frac{2\pi}{\omega}$$

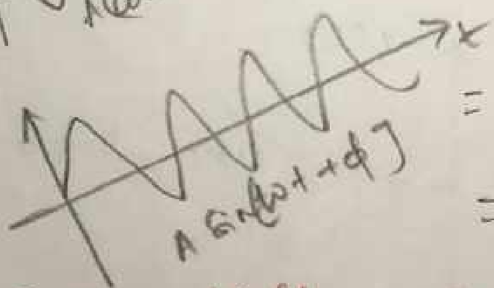
* A continuous time sinosoidal signal is periodic with a period T

$$x(t+T) = A \cos[\omega(t+T) + \phi]$$



$$= A \cos[\omega t + \omega T + \phi]$$

$$= A \cos[\omega t + \frac{2\pi}{\cancel{T}}\cancel{T} + \phi]$$



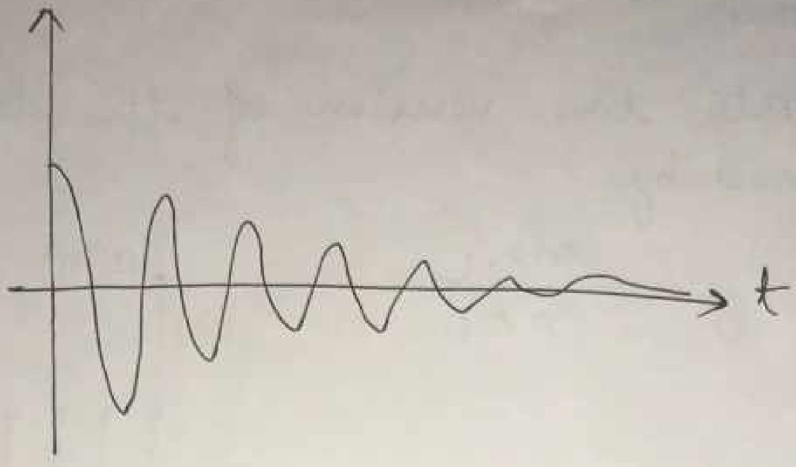
$$= A \cos[\omega t + \phi]$$

$$= x(t)$$

(6) Exponentially damped Sinosoidal Signals.

It results from multiplying Sinosoidal signal $A \sin(\omega t + \phi)$ by a real valued decaying exponential signal $e^{-\alpha t}$.

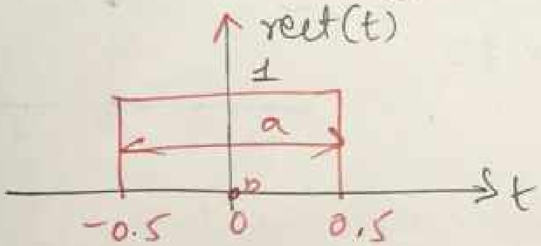
$\therefore x(t) = A e^{-\alpha t} \sin(\omega t + \phi) \quad : \alpha > 0$



(7) Pulse signals.

(i) A rectangular pulse rect(t) is defined as follows.

$$\text{rect}(t) = \begin{cases} 1, & |t| < 0.5 \\ 0, & \text{elsewhere} \end{cases}$$



height = 1 Area = 1.
width = 1

Note: The signal $x(t) = \text{rect}\left(\frac{t-b}{a}\right)$ describes a rectangular pulse of width a, centered at $t = b$.

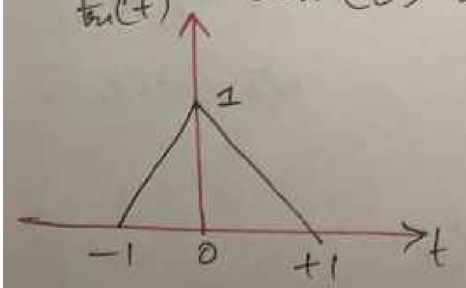
(ii) A triangular pulse tri(t) is defined as

$$\text{tri}(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$|t| > 1$

$\text{tri}\left(\frac{t}{a}\right) = 1 - \frac{|t|}{a} \quad |t| \leq a$

$\Rightarrow$



height = 1, width = 2, Area = 1.

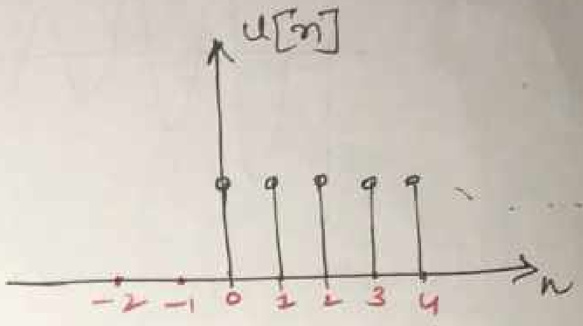
$x(t) = \text{tri}\left(\frac{t-b}{a}\right)$ describes a triangular pulse of width 2a, centered at $t = b$.

BASIC DISCRETE TIME SIGNALS.

1. STEP FUNCTION

The discrete time version of the step function $u[n]$ is defined by

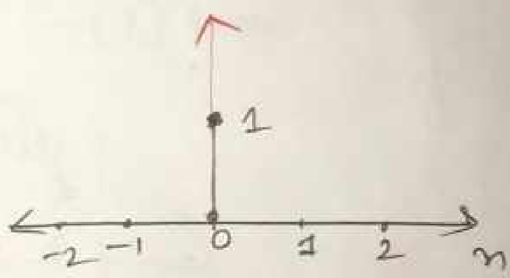
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



2. IMPULSE FUNCTION

The discrete time (signal) version of impulse function $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0 & n \neq 0 \end{cases}$$



Properties of Unit impulse function.

(1) Product Property.

check $\quad x(n)\,\delta(n-k) = x(k)\,\delta(n-k)$

(2) Shifting property.

$$\sum_{n=N_1}^{N_2} x(n)\,\delta(n-k) = \begin{cases} x(k) & N_1 \leq k \leq N_2 \\ 0 & \text{otherwise} \end{cases}$$

OR

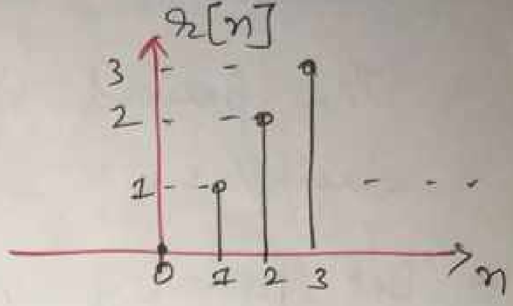
$$\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-k) = x(k).$$

(3) RAMP FUNCTION

The DT version of a ramp function is defined by

$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

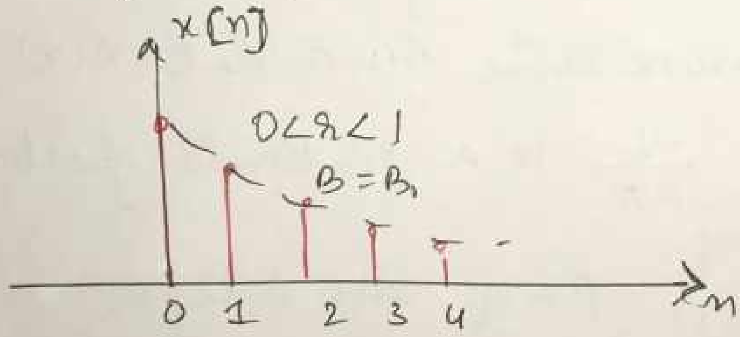
$$r(n) = n \cdot u(n)$$



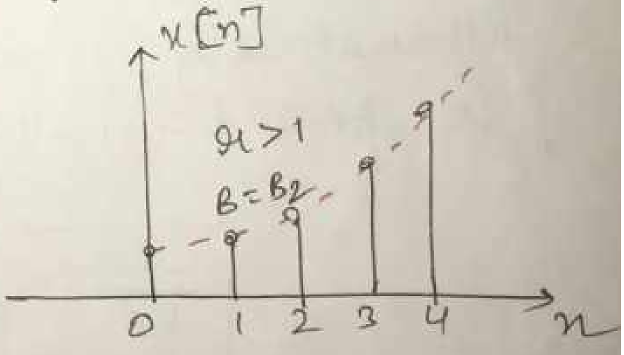
(4) EXPONENTIAL FUNCTION.

In a DT, a real exponential signal is written as $x[n] = B r^n$ where $r = e^{\alpha T}$.

* If $0 < r < 1$, $x[n] \rightarrow$ decaying exp. seqn.
  If $r > 1$, $x[n] \rightarrow$ growing —〃—.



decaying



Growing

$$x(t) = Be^{\alpha t}$$
$$B[nT) = Be^{\alpha nT}$$
$$x(n) = Br^{n}$$
$$r = e^{\alpha T}.$$

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(5) SINOSOIDAL SIGNALS.

* The DT version of a sinosoidal signal is written as $x(n) = A\cos[\Omega n + \phi]$
* The period of DT sinosoidal is measured in samples.
* Let period of $x[n]$ be N

$$x(n+N) = A\cos(\Omega n + \Omega N + \phi]$$

for $x(n)$ and $x(n+N)$ to be identical

$$\Omega N = 2\pi m$$

$$\Omega = 2\pi\left(\frac{m}{N}\right)$$

where m & N are integers

Alternatively, a discrete time sinosoidal $x(n)$ is periodic, iff, $\frac{\Omega}{2\pi}$ is a rational function

$$\boxed{\frac{\Omega}{2\pi} = \frac{p}{q}}$$ p & q are integers.

* If $\frac{\Omega}{2\pi}$ is rational, then fundamental period $N = q$ samples.
* on the otherhand, if $\frac{\Omega}{2\pi}$ is not rational, then $x[n]$ is not periodic.

(6) Exponentially damped sinusoidal signals.

The discrete - time version of the exponentially damped sinusoidal signal is described by

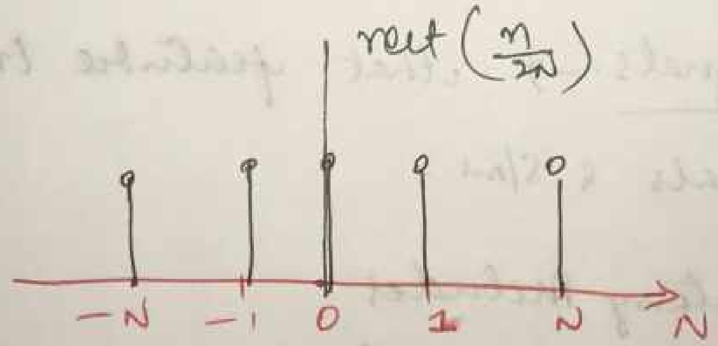
$$x[n] = Br^n \sin[\Omega n + \phi]$$

* For signal to decay exponentially with time the parameter 'r' must be in the range $0 < |r| < 1$

(7) Pulse signal

* The discrete time version of rectangular pulse $rect\left(\frac{n}{2N}\right)$ is defined as

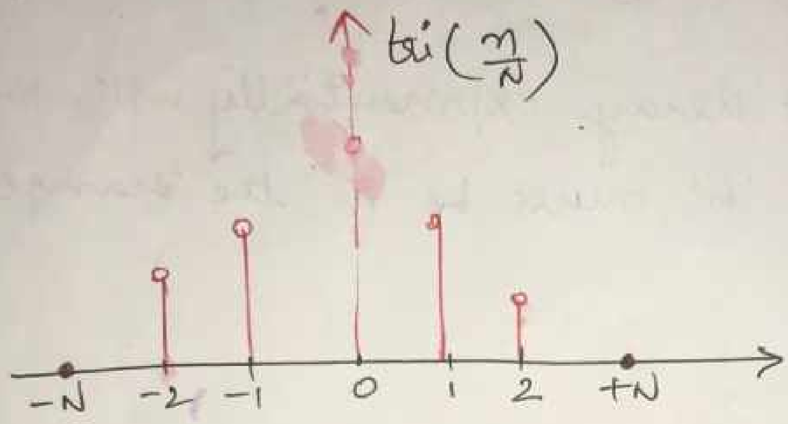
$$rect\left(\frac{n}{2N}\right) = \begin{cases} 1, & |n| \leq N \\ 0 & \text{Elsewhere} \end{cases}$$



Signal $rect\left(\frac{n}{2N}\right)$ has $2N+1$ unit samples over $-N \leq n \leq N$.

* The discrete Version of triangular pulse is defined as

$$tri\left(\frac{n}{N}\right) = \begin{cases} 1 - \frac{|n|}{N} & |n| \le N \\ 0 & \text{elsewhere.} \end{cases}$$



* The signal $tri\left(\frac{n}{N}\right)$ has $(2N+1)$ samples over $-N \le n \le N$ with $x[N]$ and $x[-N]$ being 0.

** Elementary signals. → that feature in the study of signals & s/ms.

- List of elementary includes
  - sinusoidal, exponential step, ramp, impulse

- Elementary signals serves as building blocks for the construction of more complex signals

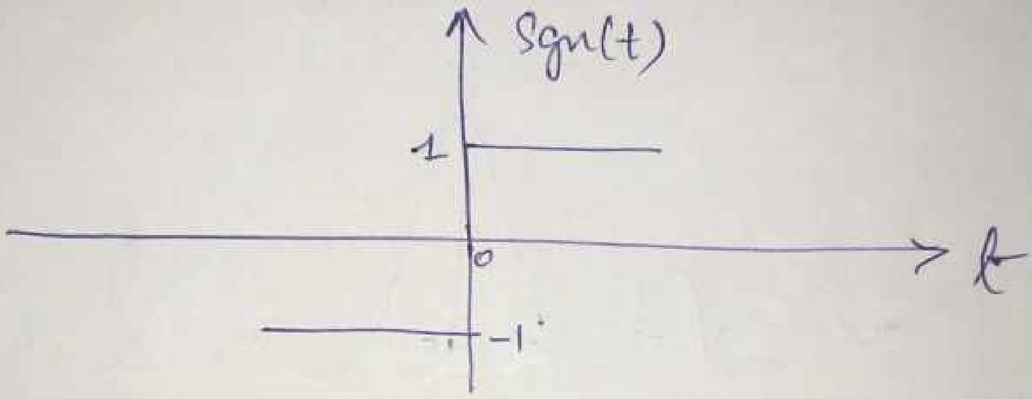
- may be used to model many physical signals that occur in nature.

(8) Signum function

CT

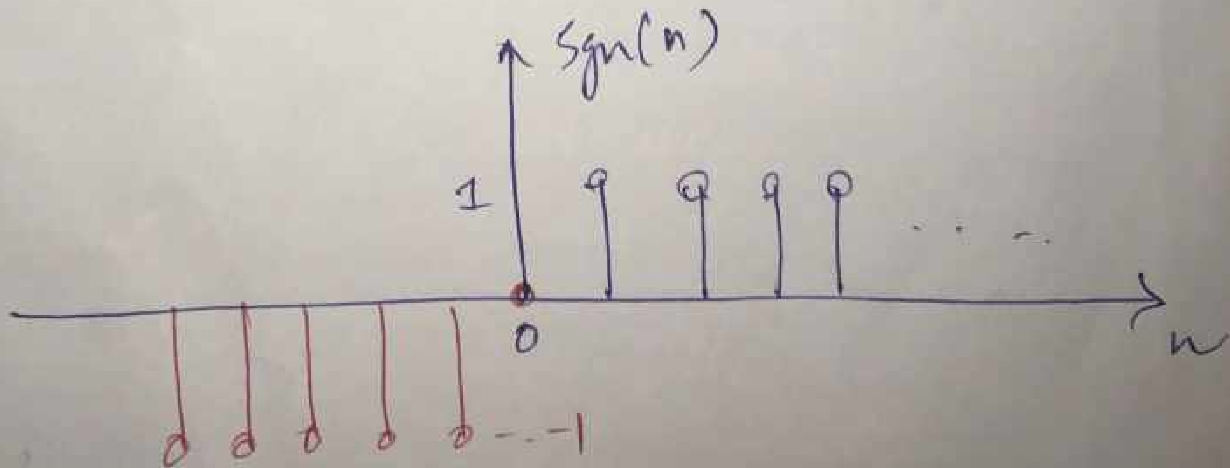
The signum function is defined as

$$\mathrm{Sgn}(t) = \begin{cases} -1 & t<0 \\ 0 & t=0 \\ 1 & t>0 \end{cases}$$



DT

$$\mathrm{Sgn}(n) = \begin{cases} 1 & n>0 \\ 0 & n=0 \\ -1 & n<0 \end{cases}$$

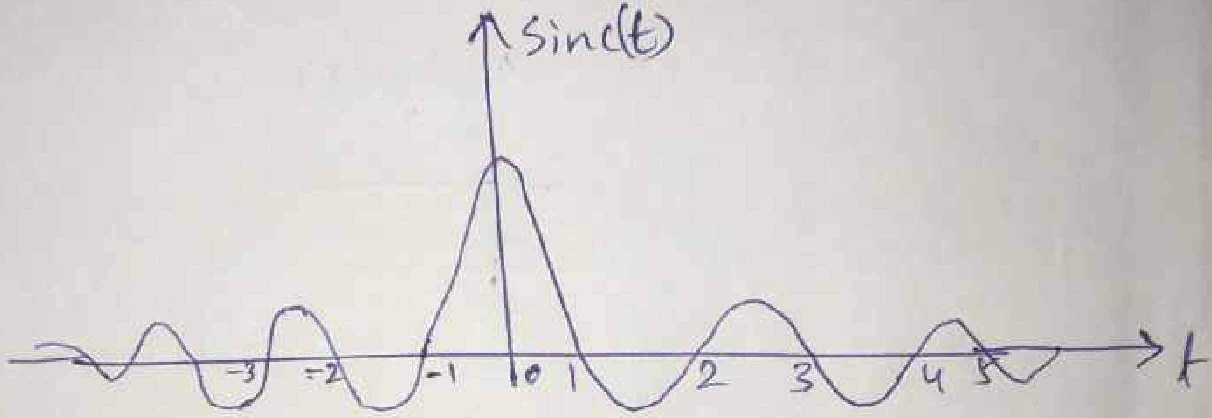


(9) Sinc function

CT.

The Sinc function is defined as

$$\text{Sinc}(t) = \frac{\sin \pi t}{\pi t}$$

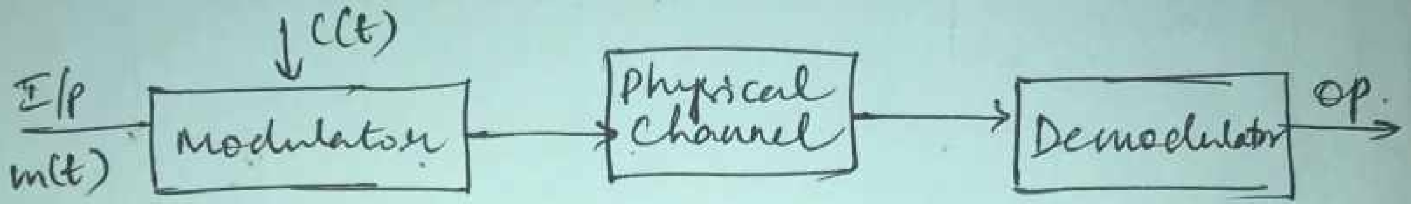


At $t=0$; $\text{sinc}(t)\Big|_{t=0} = 1$

Examples for systems.

1) communication System.

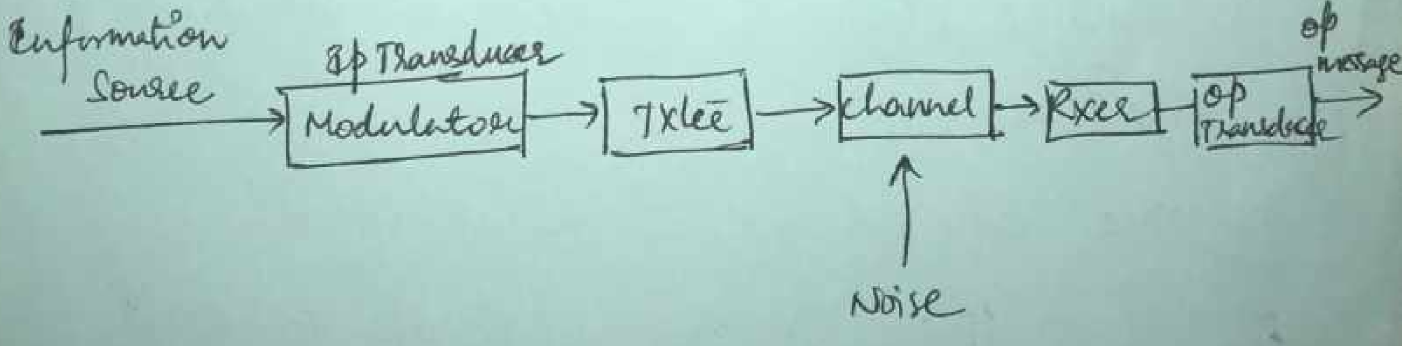
a) Analog communication System.



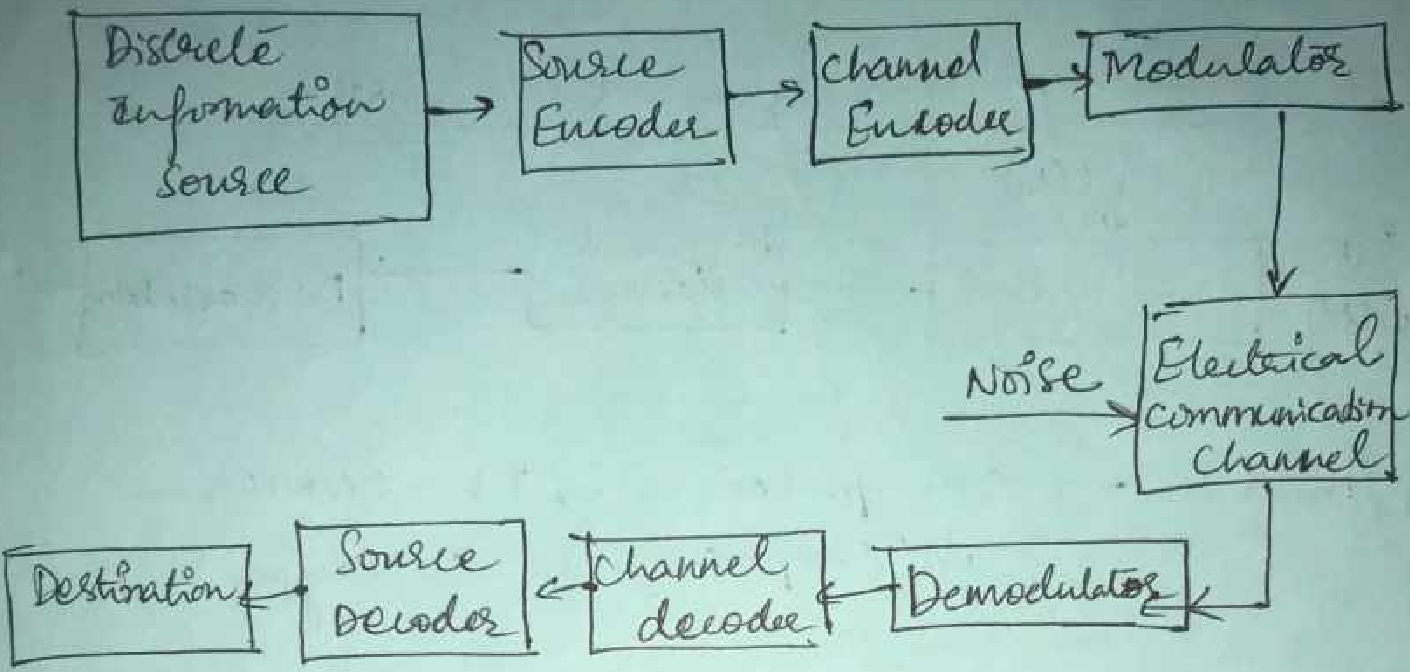
Ip/m(t) → computer, TV camera
Microphone, Teletype.

Op → Loudspeaker, CRT, TV set, Teletype
computer.

Block Diagram of communication System.



Basic block diagram of Digital communication System.



2) Control system – It is an arrangement of physical elements connected in such a manner so as to regulate, direct or command itself to achieve a certain objective.

- The control System must have
  a) ip/s, b) op/s c) arrangement to achieve this ip-op combination.

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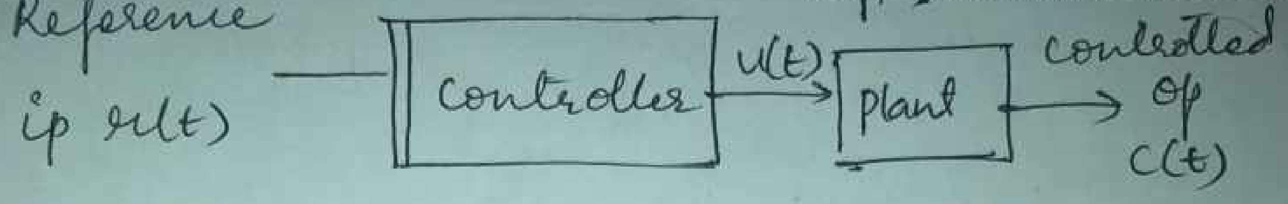
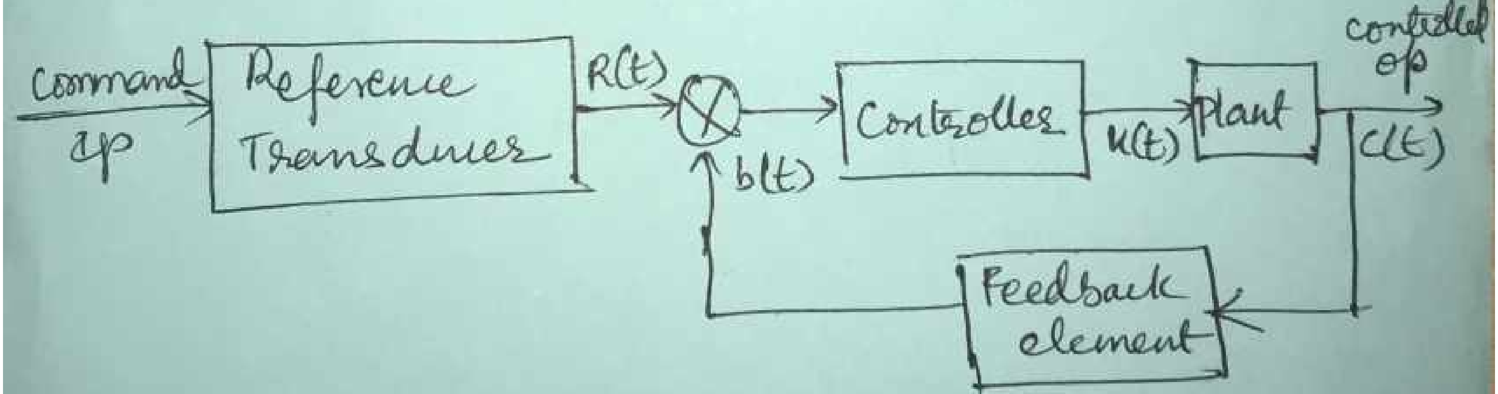
a) open loop System: Ex: Sprinkler
Traffic light controller
Room heater, Electric lift, Automatic Toaster sm.



— A system in which the control action is totally independent of the op of the system.

b) closed loop system

— A system in which the control action is somehow dependent on the op is called as closed loop system.



Ex: Human Being.
Ship Stabilization System.
Voltage Stabilizer.
Missile launching system.

Plant → The portion of the system which is to be controlled or regulated is called the plant or process.

Controller → The element of the system itself or external to the system which controls the plant or the process is called controller.

SYSTEMS VIEWED AS INTERCONNECTIONS OF OPERATIONS

* In Mathematical terms, a S/m may be viewed as an interconnection of operations that transforms an input signal into an op signal with properties different from those of the ip signal.

* The signals may be of the CT or DT variety, or a mixture of both.

* Let the overall operator $\underline{H}$ denote the action of the S/m. Then the application of a CTS $x(t)$ to the ip of the S/m yields the op signal described by

$$\boxed{y(t) = H\{x(t)\}}$$

III^ly for DTS

$$\boxed{y[n] = H\{x[n]\}}$$

Block diagram representation of operator H

$$x(t) \longrightarrow \boxed{H} \xrightarrow{y(t)}$$

Continuous Time

$$x[n] \longrightarrow \boxed{H} \xrightarrow{y[n]}$$

Discrete Time

$$x[n] \longrightarrow \boxed{S^k} \xrightarrow{x[n-k]}$$

Discrete time-shift operator S^k, operating on the DTS $x[n]$ to produce $x[n-k]$.

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1. Consider a DT S/m whose o/p signal y[n] is the average of 3 most recent values of the i/p signal x[n], as shown by

$$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

Such a S/m is referred to as a moving average s/m for 2 reasons.

* 1st, y[n] is the average of the sample values x[n], x[n-1] and x[n-2]

* 2nd, the value of y[n] changes as n moves along the discrete - time axis.

* Formulate the operator H for this S/m, hence develop a block diagram representation for it.

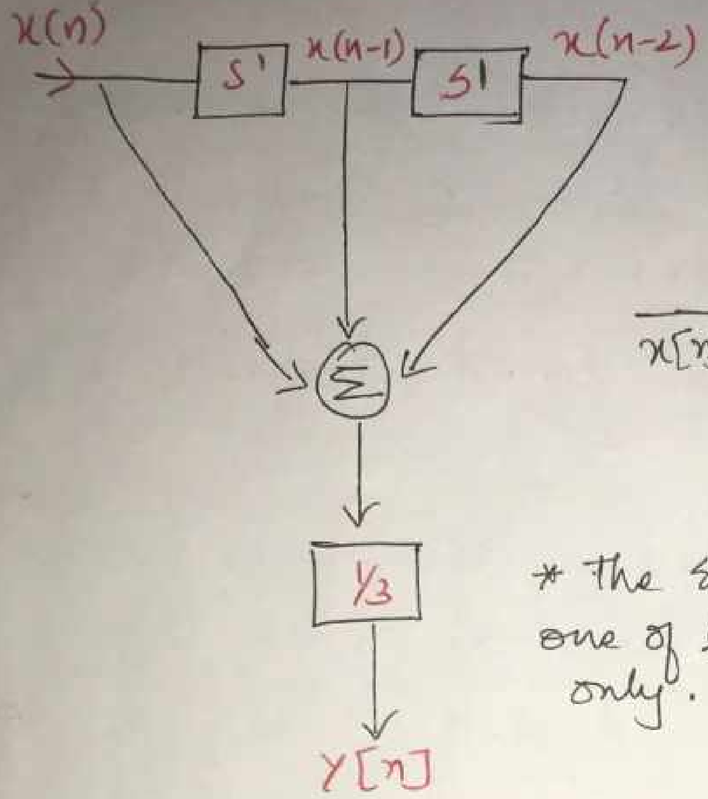
Sol^n: Let the operator S^k denote a S/m that time shifts the i/p x[n] by k time units to produce an o/p equal to x[n-k].

* Accordingly we may define the overall operator H for the moving average S/m as

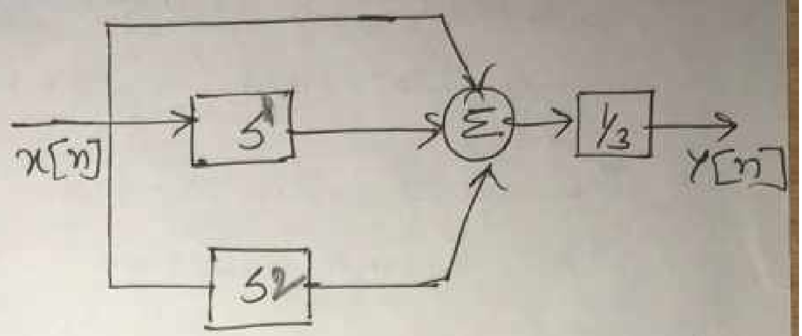
$$\boxed{H = \frac{1}{3}\left[S^0 + S^1 + S^2\right]}$$

2 different Implementations of the operator H

1) Cascade form



2) Parallel form



* the signal flows through each one of them in the forward direction only.

2) Express the operator that describes the ip-op relation

$$Y[n] = \frac{1}{3}\left[x[n+1] + x[n] + x[n-1]\right]$$ in terms of the time shift operator S.

Sol^{n}

$$H = \frac{1}{3}\left[S^{-1} + 1 + S^{1}\right]$$

PROPERTIES OF SYSTEM.

* The properties of a S/m describe the characteristics of the operator H representing the S/m.

1. **STABILITY** : A S/m is said to be bounded ip bounded op (BIBO) stable if and only if every bounded ip results in a bounded op.

* The op of such a S/m doesnot diverge if the ip doesnot diverge.

* To put the condition for BIBO stability on a formal basis, consider a CT S/m whose ip-op relation is given by $\boxed{y(t) = H\{x(t)\}}$.

* The operator H is BIBO stable if the op signal y(t) satisfies the condition

$$\boxed{|y(t)| \le M_y < \infty \text{ for all } t.}$$

whenever the ip signal x(t) satisfy the condition

$$\boxed{|x(t)| \le M_x < \infty \text{ for all } t}$$

Both M_x and M_y represent some finite positive integers/numbers.

III^ly for DTS,

$$\boxed{\begin{aligned} |y[n]| &\le M_y < \infty, \quad \forall n \\ |x[n]| &\le M_x < \infty, \quad \forall n \end{aligned}}$$

Problems

① ST the moving average S/m
$$y[n] = \frac{1}{3}\left[x[n] + x[n-1] + x[n-2]\right]$$ is BIBO ?

Sol^n

Assume that
$$|x[n]| \le M_x < \infty \qquad \text{for all } n$$

Using given i/p - o/p relation,
$$y[n] = \frac{1}{3}\left[\ x[n] + x[n-1] + x[n-2]\right]$$

We may write
$$\begin{aligned}|y[n]| &\le \frac{1}{3}\left|x[n] + x[n-1] + x[n-2]\right| \\ &\le \frac{1}{3}\left(|x[n]| + |x[n-1]| + |x[n-2]|\right) \\ &\le \frac{1}{3}\left(M_x + M_x + M_x\right) \\ &\le M_x.\end{aligned}$$

Hence the absolute value of the o/p signal $y[n]$ is always less than the maximum absolute value of the i/p signal $x[n]$ for all n, which shows that the moving-average S/m is stable.

② Same as problem ① for
$$y[n] = \frac{1}{3}\left[x(n+1) + x[n] + x[n-1]\right]$$

③ Consider a DT S/m whose i/p - o/p relation is defined by $y[n] = r^n x[n]$ where $r > 1$.

ST the S/m is unstable.

Sol^n. Assume that the i/p signal $x[n]$ satisfies the condition
$$|x[n]| < M_x < \infty \qquad \forall n$$

We then find that

$$|y[n]| \leq |r^n x[n]|$$

$$\leq |r^n| \cdot \underbrace{|x[n]|}_{M_x}$$

with $r > 1$, the multiplying factor r^n diverges for increasing n. Accordingly, the condition that the ip signal is bounded is not sufficient to guarantee a bounded op signal, and so the s/m is unstable.

2. MEMORY: A s/m is said to possess memory if its op signal depends on past values of the ip signal. Ex: Inductor $i(t) = \frac{1}{L}\int_{-\infty}^{\infty} v(\tau)\,d\tau$ [i @ t depends on past values of $v(t)$]

* A s/m is said to be memoryless if its op signal depends only on the present value of the ip signal.

Ex: Resistor $i(t) = \frac{1}{R} v(t)$

(i) $y[n] = \frac{1}{3}\left[x[n] + x[n-1] + x[n-2]\right]$ — the moving average s/m has memory, since the value of the op signal $y[n]$ at time n depends on the present and 2 past values of the ip signal $x[n]$.

(ii) $y[n] = x^2[n]$ → is memoryless, since the value of the op signal $y[n]$ at time n depends only on the present value of the ip signal $x[n]$.

(1) How far does the memory of the moving average s/m described by the ip-op relation

$$y[n] = \frac{1}{3}\left[x[n] + x[n-1] + x[n-2]\right]$$ extend into the past?

Sol^{n}: 2 time Units $\quad x[n-2] \xleftarrow{2} x[n-1] \xleftarrow{1} x[n]$

(2) $$V(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau$$

Memory extends from t to the ∞ past

3. CAUSALITY: A s/m is said to be causal if the present value of the op signal depends only on the present and/or past values of the ip signal.

* The op signal of a non-causal s/m depends on future values of the ip signal.

Ex: (i) $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right) \rightarrow$ Causal

(ii) $y[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right) \rightarrow$ Non causal.

4. INVERTIBILITY: A s/m is said to invertible if the ip of the s/m can be recovered from the s/m op.

$$\xrightarrow{x(t)} \boxed{H} \xrightarrow{y(t)} \boxed{H^{-1}} \xrightarrow{x(t)}$$

NOTATION OF S/M INVERTIBILITY

- Let the operator H represents a CT s/m with the ip signal $x(t)$ producing the o/p signal $y(t)$.
- Let the o/p signal $y(t)$ be applied to a second continuous time s/m represented by the operator H^{-1}

The output signal of the second S/m is defined by

$$H^{-1}\{y(t)\} = H^{-1}\{H(x(t))\}$$
$$= H^{-1}H\{x(t)\}$$

where we have made use of the fact that 2 operators H and H^{-1} connected in cascade are equivalent to a single operator $H^{-1}H$.

$$x(t) \to \boxed{HH^{-1}} \to x(t)$$

* For this op signal to equal the original ip signal $x(t)$, we require that

$$\boxed{H^{-1}H = I} \to (1)$$

where $I \to$ Identity operator.

$H^{-1} \to$ Inverse operator.

* The op of a S/m described by the identity operator is exactly equal to the ip.

* Eq (1) is the condition that the new operator H^{-1} must satisfy in relation to the given operator H for the original ip signal $x(t)$ to be recovered from $y(t)$.

* The associated S/m is called the inverse s/m.

* H^{-1} is not the reciprocal of the operator H. Superscript (-1) is intended to be merely a flag indicating inverse.

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* Problem of finding the inverse of a given Slm is difficult one.
* In any event, a Slm is not invertible unless distinct ips applied to the Slm produce distinct ops. ie, there must be a one-to-one mapping b/w ip and op signals for a Slm to be invertible.

* IIIly, the identical conditions must hold for a Discrete time system to be invertible.

① Consider the time-shift Slm described by the ip-op relation.

$$y(t) = x(t-t_0) = S^{t_0}\{x(t)\}$$

where S^{t_0} represents a time shift of t_0 seconds. Find the inverse of this Slm.

Solⁿ The inverse of a time shift of t_0 seconds is a timeshift of $-t_0$ seconds.

* We may represent the time shift of $-t_0$ by the operator S^{-t_0}.
* Thus by applying S^{-t_0} to the op signal of the given time-shift Slm

$$S^{-t_0}\{y(t)\} = S^{-t_0}\{S^{t_0}\{x(t)\}\}$$

$$= S^{-t_0}S^{t_0}\{x(t)\}$$

* For this op signal to equal the original ip signal $x(t)$, we require that

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$$\boxed{S^{-t_0} S^{t_0} = I}$$

which is in perfect accord with the condition for invertibility.

2) S.T a square law s/m described by the i/p-o/p relation
$$y(t) = x^2(t)$$ is not invertible?

Sol^n: * We note that the square-law S/m violates a necessary condition for invertibility, which postulates that distinct i/ps must produce distinct o/ps.

* Specifically, the distinct inputs $x(t)$ & $-x(t)$ produce the same o/p $y(t)$.

* Accordingly, the square-law S/m is not invertible.

5. TIME INVARIANCE.

* A system is said to be time invariant, if a time-delay or time advance of the i/p signal leads to an identical time-shift in the o/p signal.

* This implies that a time-invariant S/m responds identically no matter when the signal is applied.

* The characteristics of a time-invariant S/m donot change with time.

* otherwise the S/m is said to be time-variant.

→ Consider a CT s/m whose i/p-o/p relation is given by

$$y(t) = H\{x(t)\}$$

* Suppose the i/p signal $x(t)$ is shifted in time by t_0 seconds, resulting in the new i/p $x(t-t_0)$. This operation is described by

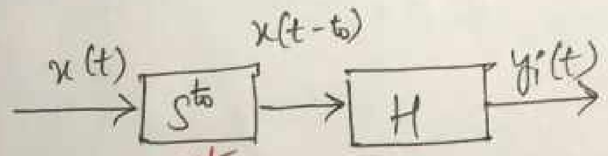
$$x(t-t_0) = S^{t_0}\{x(t)\}$$

where S^{t_0} → time shift equal to t_0 seconds.

→ Let $y_i(t)$ denote the o/p signal of the s/m produced in response to the time-shifted i/p $x(t-t_0)$.

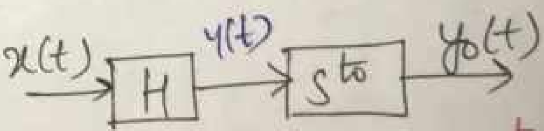
$$y_i(t) = H\{x(t-t_0)\}$$
$$= H\{S^{t_0}\{x(t)\}\}$$
$$= HS^{t_0}\{x(t)\}$$

which is represented by the BD shown (BD1)



Time-shift operator S^{t_0} preceding operator H

BD①



Time shift operator S^{t_0} following operator H.

BD②

* ~~The s/m is time invariant if the o/ps $y_i(t)$ and $y_0(t)$ are equal~~

- Suppose $y_o(t)$ represents the o/p of the original s/m shifted in time by t_0 seconds.

$$y_o(t) = S^{t_0}\{y(t)\}$$
$$= S^{t_0}\{H\{x(t)\}\}$$
$$= S^{t_0} H\{x(t)\}$$

which is represented by the Block diagram BD2.

→ The s/m is time invariant if the o/ps $y_i(t)$ & $y_o(t)$ are equal for any identical i/p signal $x(t)$.

Hence we require $\boxed{HS^{t_0} = S^{t_0}H}$

→ That is for a s/m described by the operator H to be time invariant, the s/m operator H and the time-shift operator S^{t_0} must commute with each other for all t_0.

→ A similar relation must hold for a DT system to be time invariant.

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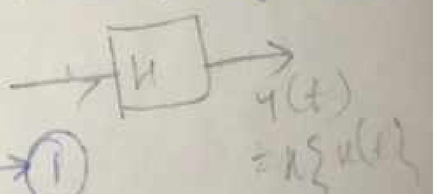
6. Linearity

* A S/m is said to be linear if it satisfies the principle of superposition. That is the response of a linear system to a weighted sum of i/p signals is equal to the same weighted sum of o/p signals, each o/p signal being associated with a particular i/p signal acting on the S/m independently of all the other i/p signals.

* A S/m that violates the principle of superposition is called nonlinear S/m.

* Let the operator H represent a continuous-Time S/m. Let the signal applied to the S/m i/p be defined by the weighted sum



$$\boxed{x(t) = \sum_{i=1}^{N} a_i x_i(t)} \longrightarrow (1)$$

where $x_1(t), x_2(t) \ldots x_N(t)$ denote a set of i/p signals, and $a_1, a_2, \ldots a_N$ denote the corresponding weighting factors. The resulting o/p signal is defined as,

$$y(t) = H\{x(t)\}$$

$$\boxed{y(t) = H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\}} \longrightarrow (1a)$$

* If the S/m is linear, we may express the o/p signal y(t) of the S/m as

$$\boxed{y(t) = \sum_{i=1}^{N} a_i y_i(t)} \longrightarrow (2)$$

Where $y_i(t)$ is the o/p of the S/m in response to the i/p $x_i(t)$ acting alone, i.e,

$$\boxed{y_i(t) = H\{x_i(t)\}.} \longrightarrow (2a)$$

The weighted sum of eqn (2) describing the o/p signal $y(t)$ is of the same mathematical form as that of eqn (1), describing the i/p signal $x(t)$.

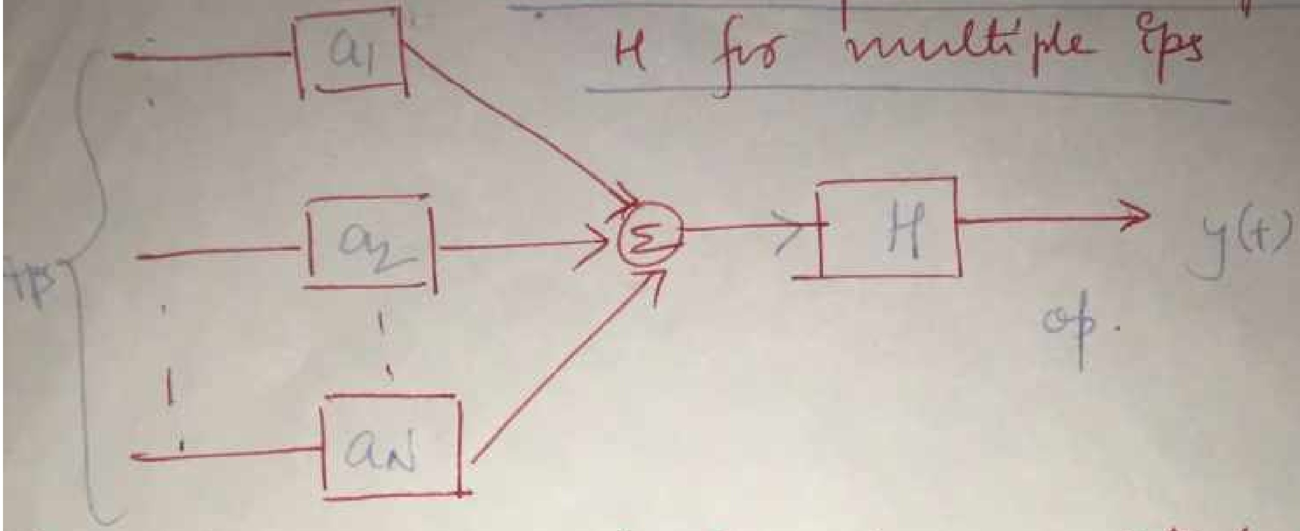
$$\boxed{y(t) = \sum_{i=1}^{N} a_i H\{x_i(t)\}} \longrightarrow (3)$$

In order to write eqn (3) in the same form as 1(a), the S/m operation described by H must commute with the Summation and amplitude Scaling in eqn (3).

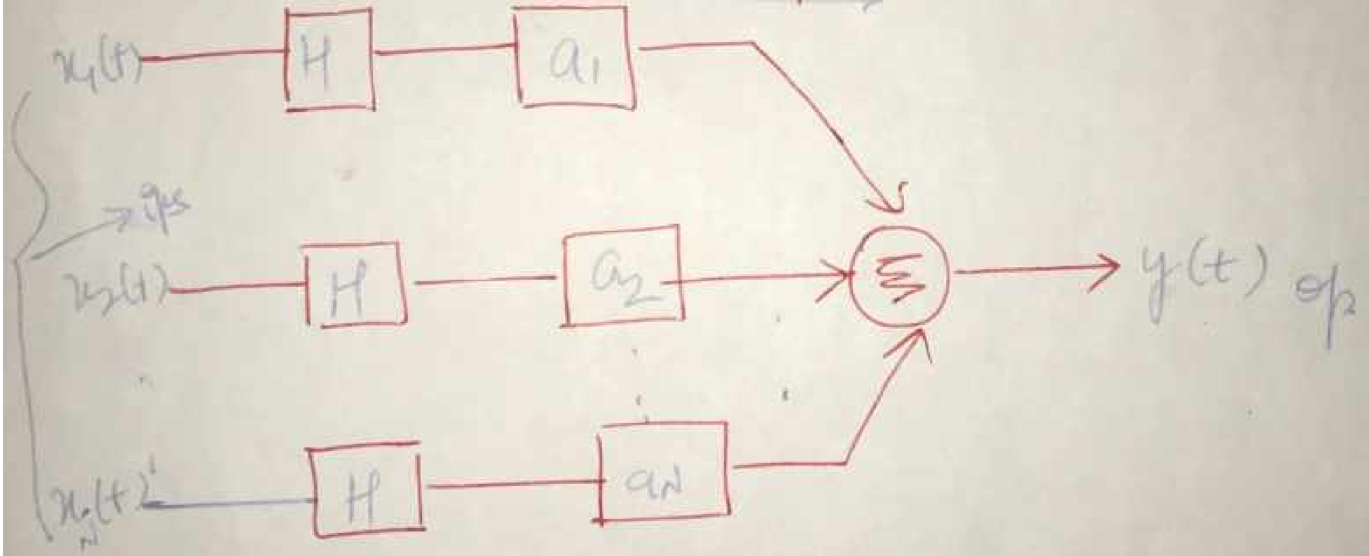
— Or eqns (2a) and (3) represent the mathematical statement of the principle of superposition.

— $||^{ly}$ for DT System.

(1) Combined operation of amplitude scaling & summation precedes the operator H for multiple ips

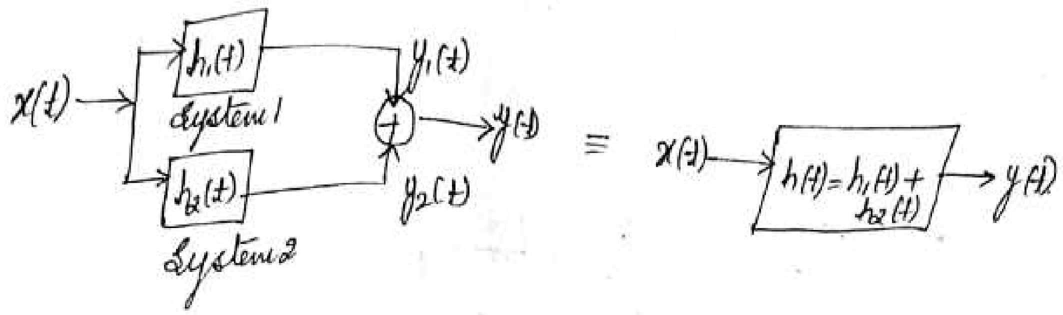


(2) The operator H precedes amplitude scaling for each ip.

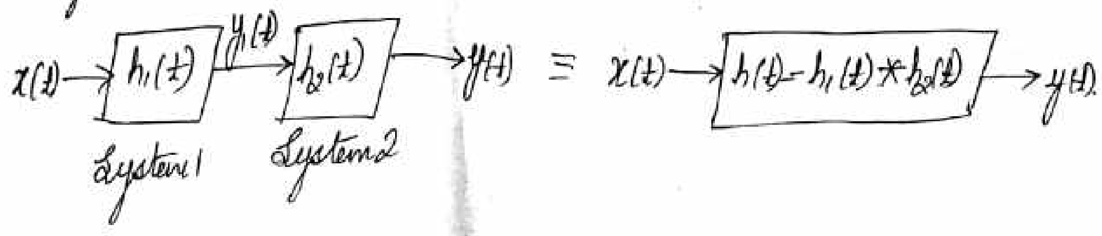


Impulse response of Interconnected LTI systems.

1) Systems in parallel



2. Systems in Cascade



$\boxed{\text{Systems in parallel.}}$

$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

$$y(t) = y_1(t) + y_2(t)$$

$$y(t) = \underline{x(t) * h_1(t)} + \underline{x(t) * h_2(t)}$$

$$= x(t) * \underbrace{[h_1(t) + h_2(t)]}_{h(t)}$$

$$y(t) = x(t) * h(t)$$

$\boxed{\text{Systems in cascade.}}$

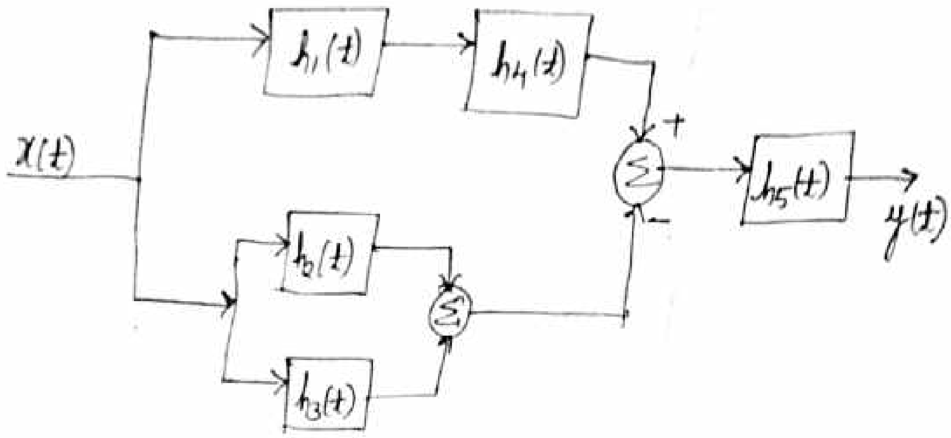
$$y(t) = y_1(t) * h_2(t)$$

$$= x(t) * \underbrace{h_1(t) * h_2(t)}_{h(t)}$$

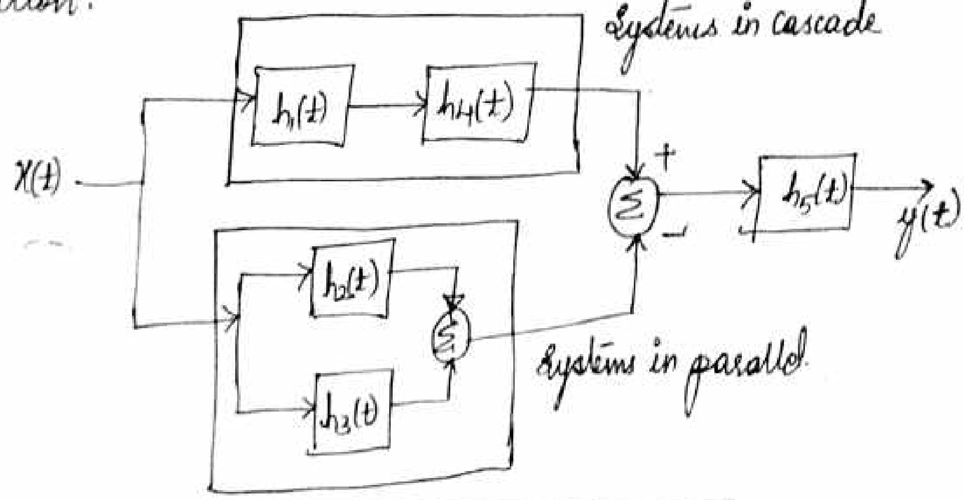
$$y(t) = x(t) * h(t)$$

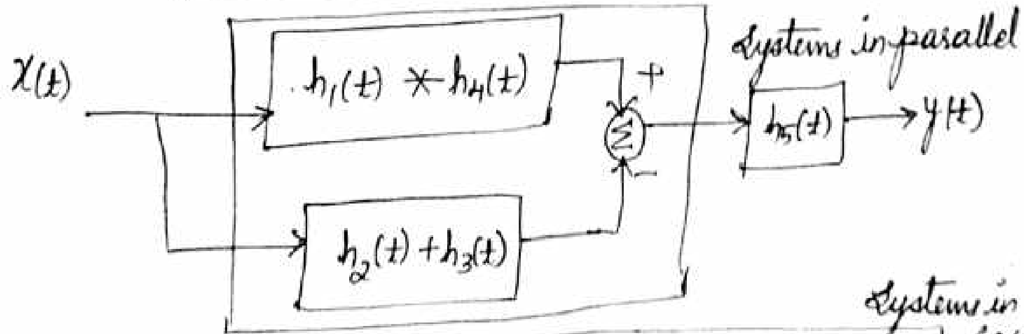
Example: Find the expression for the impulse response relating the input $x(t) / x[n]$ to the output $y(t) / y[n]$ in terms of the impulse responses of each subsystem for the LTI systems described

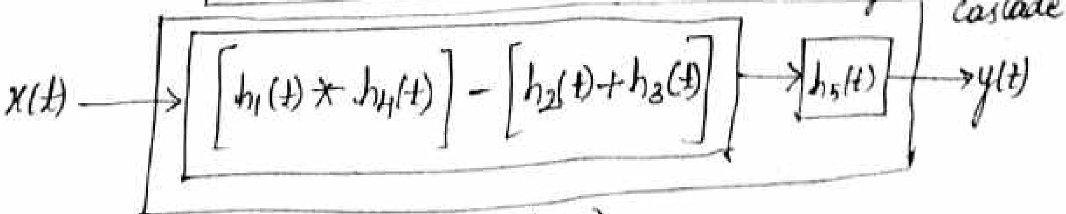
1)

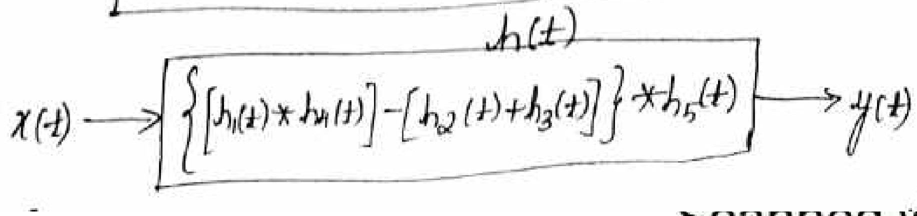


Solution:

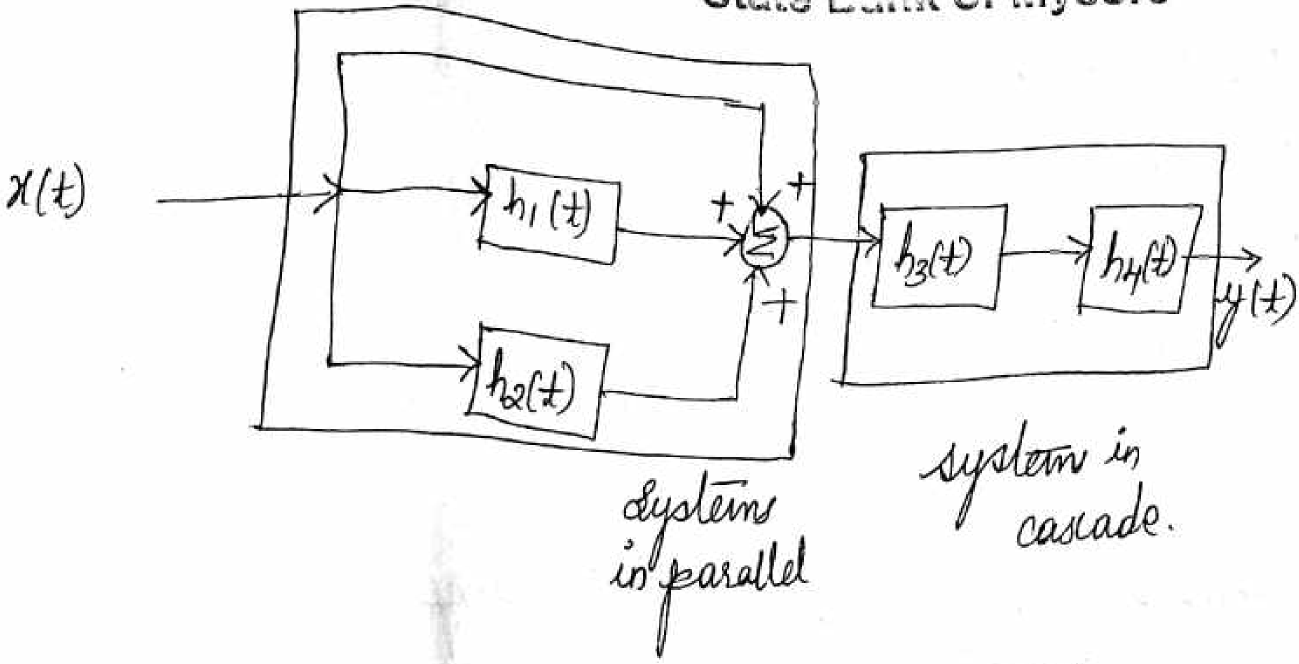


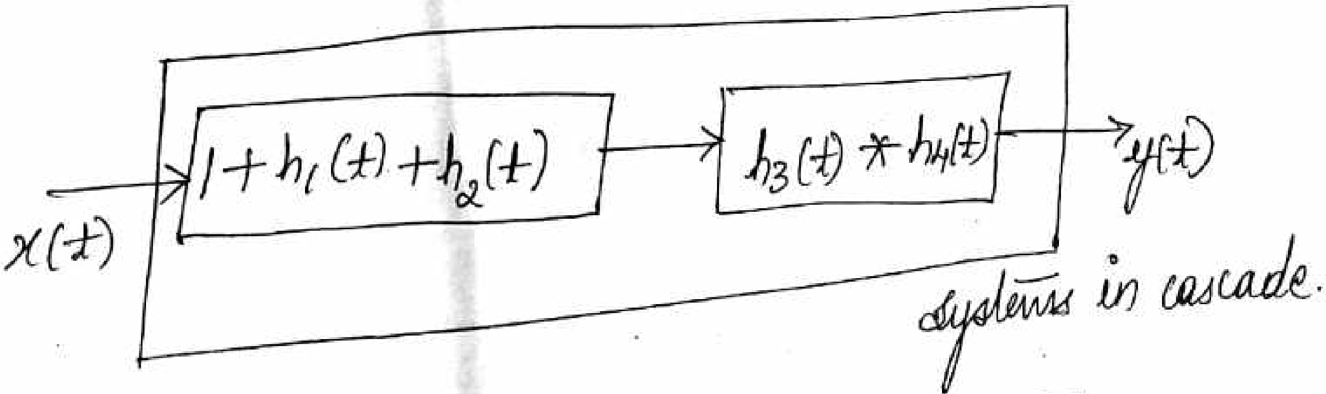


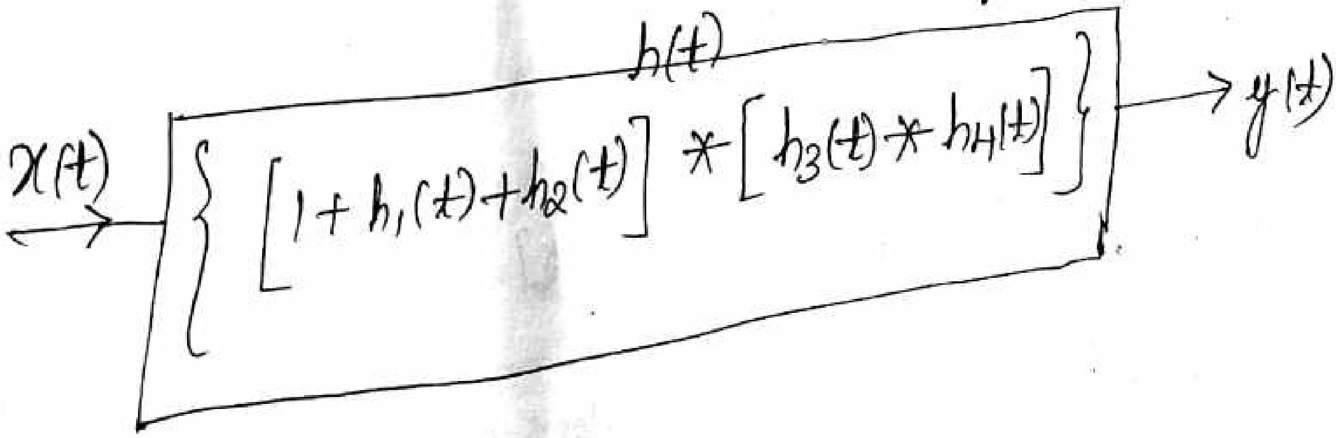




(2)


x(t)
h1(t)
h2(t)
Σ
+
+
+
h3(t)
h4(t)
y(t)
systems in parallel
system in cascade.


x(t)
1 + h1(t) + h2(t)
h3(t) * h4(t)
y(t)
systems in cascade.


h(t)
x(t)
{ [1 + h1(t) + h2(t)] * [h3(t) * h4(t)] }
y(t)

Properties of LTI system.

- Memoryless
- Causality
- Stability
- Investibility

Memoryless:

Consider a LTI system with Impulse response $h[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

using commutative property,

$$y[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Expanding the expression for few terms

$$y[n] = \cdots\cdots + h[-2]\,x[n+2] + h[-1]\,x[n+1] + h[0]\,x[n] + h[1]\,x[n-1] + h[2]\,x[n-2] + \cdots$$

For LTI system to be memoryless, $y[n]$ must depend only on $x[n]$ and cannot depend on $x[n-k]$ for $k \neq 0$

$$\boxed{\begin{aligned} h[k] &= c \cdot \delta[k] \\ &\& \\ h[k] &= 0 \quad \text{for } k \neq 0 \end{aligned}}$$

Analogous to discrete-time system, a continous time system is memoryless if and only if

$$h(t) = c\,\delta(t)$$

Causal : The output of causal system depends only on past or present values of the input.
Consider,

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$y[n] = \cdots\cdots + h[-2]x[n+2] + h[-1]\, x[n+1] + h[0]\, x[n] + h[1]\, x[n-1] + h[2]x[n-2] + \cdots\cdots$$

Here past and present values of i/p $x[n], x[n-1], \cdots$ are associated with indices $k \geq 0$ in the convolution sum, while future values are associated with $k < 0$.

for the system to be causal,

$$\boxed{h[k] = 0, \quad \text{for } k < 0}$$

$$h(\tau) = 0 \quad \text{for} \quad \tau < 0$$

Stable.

The s/m is BIBO stable if the o/p is bounded for every bounded i/p

$$y[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Let $x[n] = M_x < \infty$ for stability $|y[n]| \leq M_y < \infty$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h(k)\, M_x\right| = |y[n]| = \sum_{k=-\infty}^{\infty} |h[k]| \underbrace{|x[n-k]|}_{M_x}$$

for the system ~~should~~ to be stable

$$y[n] < \infty$$

so,

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty}$$

Continous time system

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty}$$

Invertible Systems and Deconvolution

- A system is invertible if the input to the system can be recovered from the output

- This implies that there should be an inverse system that takes the output of the original system as its input and produces the input of the original system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \quad = y(t) = x(t) * h(t).$$

- Requires an LTI system connected in cascade to the original s/m with Impulse response $h^{-1}(t)$

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{\;y(t)\;} \boxed{h^{-1}(t)} \longrightarrow x(t)$$

- The process of recovering $x(t)$ from $y(t)$
  = deconvolution = reverses / undo the convolution operation.

$$x(t) * \underbrace{\left[h(t) * h^{-1}(t)\right]}_{\delta(t)} = x(t)$$

$$x(t) * \delta(t) = x(t)$$

For discrete time system,

$$h[n] * h^{-1}[n] = \delta[n].$$

Invertible Systems and Deconvolution

- A system is invertible if the input to the system can be recovered from the output.
- This implies that there should be an inverse system that takes the output of the original system as its input and produces the input of the original system.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad = \quad y(t) = x(t) * h(t).$$

- Requires an LTI system connected in cascade to the original s/m with impulse response $h^{-1}(t)$

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{\;y(t)\;} \boxed{h^{-1}(t)} \longrightarrow x(t)$$

- The process of recovering $x(t)$ from $y(t)$
  = deconvolution = reverses / undo the convolution operation.

$$x(t) * \underbrace{\left[h(t) * h^{-1}(t)\right]}_{\delta(t)} = x(t)$$

$$x(t) * \delta(t) = x(t)$$

For discrete time system,

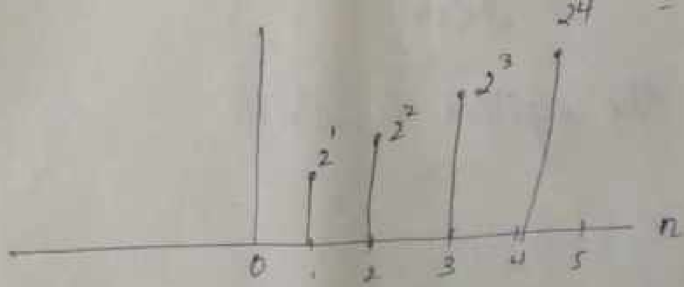
$$h[n] * h^{-1}[n] = \delta[n].$$

Recap

Property	Condition	
Memoryless	$h[k] = c \cdot \delta[k]$. $h[k] = 0$, $k \neq 0$	$h(\tau) = c \cdot \delta(\tau)$ $h(\tau) = 0$ $\tau \neq 0$
Causal	$h[k] = 0$, $k < 0$	$h(\tau) = 0$, $\tau < 0$
Stability	$\left\| \sum_{k=-\infty}^{\infty} h(k) \right\| < \infty$	$\int_{-\infty}^{\infty} h(\tau) d\tau < \infty$
Invertible	$h[n] * h^{-1}[n] = \delta[n]$	$h(t) * h^{-1}(t) = \delta(t)$

Example 1

$h[n] = 2^n u[n-1]$



Memoryless : $h[n] \neq 0$ $n \neq 0$

has memory.

Causal : $h[n] = 0$ $n < 0$

the system is causal.

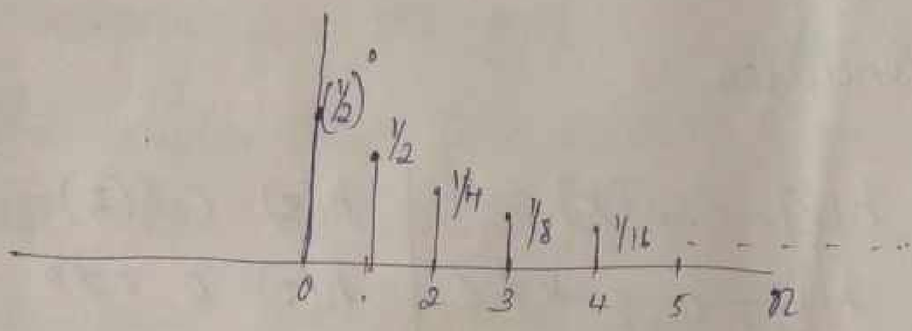
Stability : $\sum_{k=-\infty}^{\infty} h[k] = 2^1 + 2^2 + 2^3 + \dots + 2^{\infty}$

$= \infty$ unstable

the system is unstable

Example 2:

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$



Memoryless: $h[n] \neq 0$; $n \neq 0$

the system has memory.

Causal: $h[n] = 0$; $n < 0$

the system is causal.

Stability:
$$\sum_{n=-\infty}^{\infty} h[n] = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2}}$$

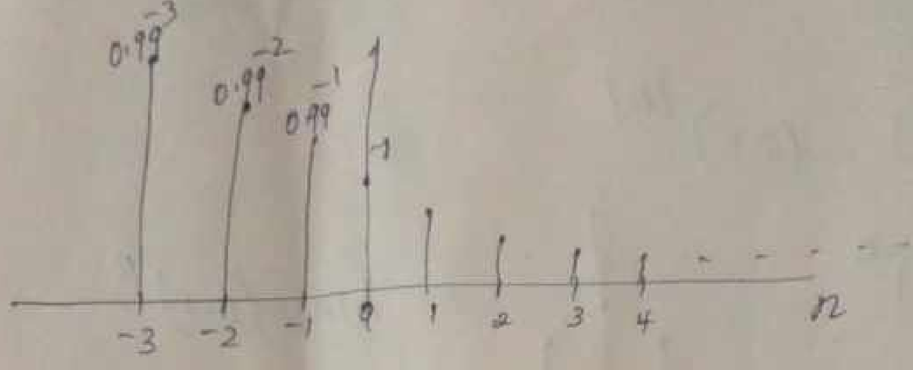
$$\left\langle \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right.$$

$$= 2 < \infty$$

the system is stable.

Example 3 :

$$h[n] = (0.99)^n u[n+3]$$



memoryless : $h[n] \neq 0 \;;\; n \neq 0$

the system has memory.

Causality : $h[n] \neq 0 \;:\; n < 0$

the system is not causal.

Stability $\sum_{n=-3}^{\infty} (0.99)^n$

let $n+3 = l \Rightarrow n = l-3$

$n = -3 \quad l = 0$

$n = \infty \quad l = \infty$

$$\sum_{l=0}^{\infty} (0.99)^{l-3}$$

$$= (0.99)^{-3} \boxed{\sum_{l=0}^{\infty} (0.99)^{l}}$$

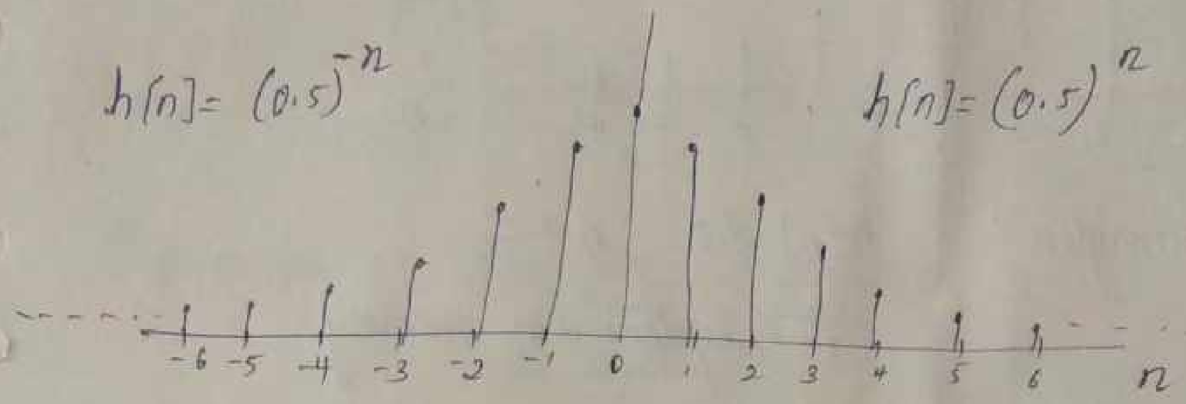
$$= (0.99)^{-3} \times \frac{1}{1-0.99}$$

$$= 103.06 < \infty$$

the system is stable

Example 4:

$$h[n] = (0.5)^{|n|}$$



Memoryless: $h[n] \neq 0 \; : n \neq 0$

the system has memory.

Causal : $h[n] \neq 0 \; : n < 0$

the system is non causal

Stability

$$\sum_{n=-\infty}^{\infty} (0.5)^{|n|}$$

$$= \sum_{n=-\infty}^{-1} (0.5)^{-n} + \sum_{n=0}^{\infty} (0.5)^{n}$$

$$= \sum_{n=\infty}^{1} (0.5)^{n} + \sum_{n=0}^{\infty} (0.5)^{n} \qquad \left\langle \sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \right.$$

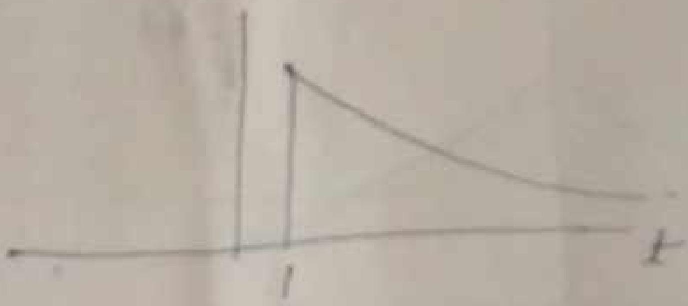
$$= \frac{0.5}{1-0.5} + \frac{1}{1-0.5}$$

$$= 3 < \infty$$

the system is stable

Example 5:

$$h(t) = e^{-3t} u(t-1)$$



Memoryless: $h \neq 0 ; t \neq 0$

the system has memory

Causal: $h(t) = 0 ; t < 0$

the system is causal

Stability $\int_{-\infty}^{\infty} h(\tau)\, d\tau$

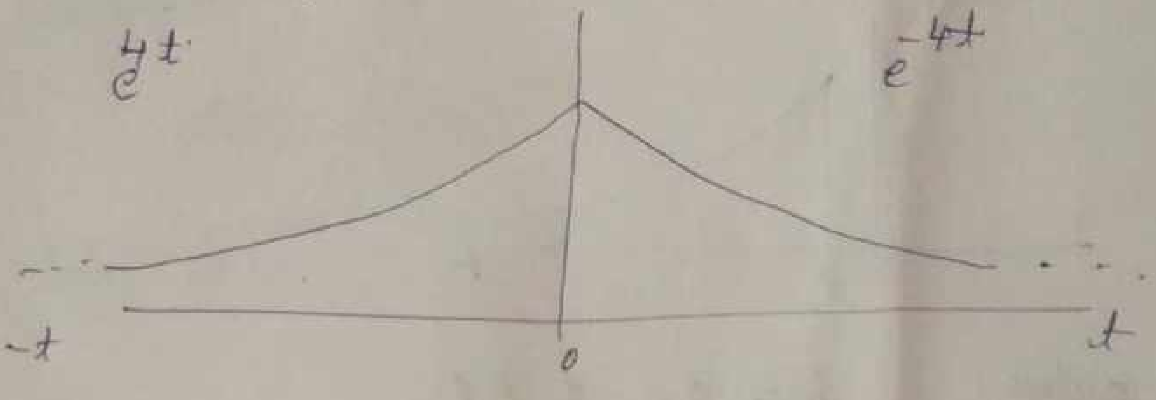
$$= \int_{1}^{\infty} e^{-3t}\, dt = \left.\frac{e^{-3t}}{-3}\right]_{1}^{\infty}$$

$$= \frac{e^{-3}}{3} < \infty$$

the system is stable

Example 6.

$$h(t) = e^{-4|t|}$$



Memoryless : $h(t) \neq 0 \quad t \neq 0$

It has memory.

Causal $h(t) \neq 0 \quad t < 0$

the system is non causal.

Stability $\int_{-\infty}^{\infty} e^{-4|t|} dt$

$$= \int_{-\infty}^{0} e^{4t} dt + \int_{0}^{\infty} e^{-4t} dt.$$

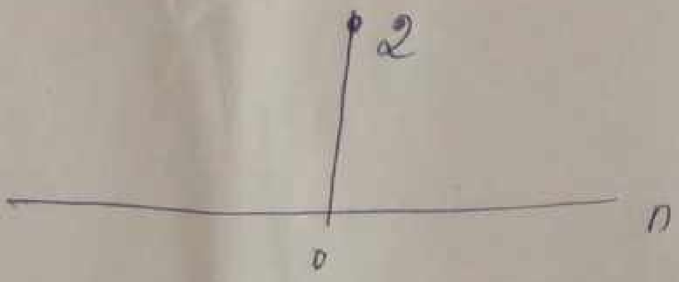
$$= \left[\frac{e^{4t}}{4}\right]_{-\infty}^{0} + \left[\frac{e^{-4t}}{-4}\right]_{0}^{\infty}$$

$$= \frac{1}{2} < \infty$$

the system is stable

Example 7:

$h[n] = 2u[n] - 2u[n-1]$



Memoryless $\quad h[n] = 0 \quad n \neq 0$

the system is memoryless

Causal $\quad h[n] = 0 \; ; \; n < 0$

the system is causal.

Stability $\quad 2 < \infty$

the system is stable .

the system is

Assignment:

$h[n] = \delta[n] + \sin n\pi$

$h(t) = 3\,\delta(t)$

$h[n] = 4^{-n}\, u[2-n]$

$h[n] = n\left(\frac{1}{2}\right)^{n} u[n]$

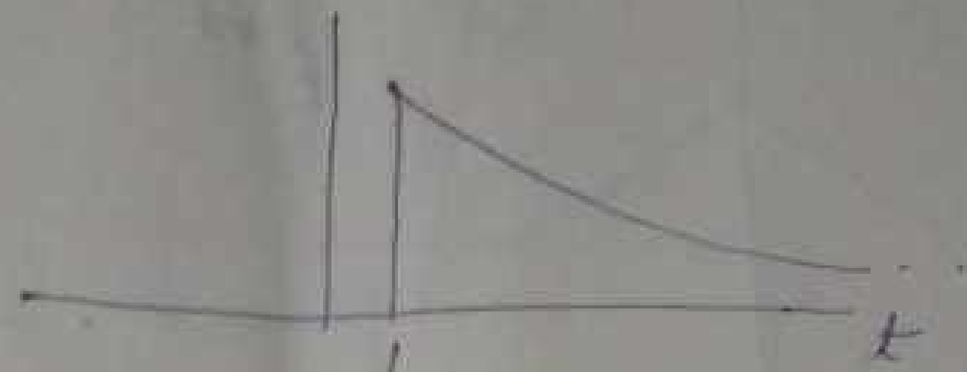
$h(t) = e^{t}\, u(-t-1)$

Properties

Property	Condition	
Memoryless	$h[k] = c \cdot \delta[k]$. $h[k] = 0$, $k \neq 0$	$h(\tau) = c \cdot \delta(\tau)$ $h(\tau) = 0$ $\tau \neq 0$
Causal	$h[k] = 0$, $k < 0$	$h(\tau) = 0$, $\tau < 0$
Stability	$\left\lvert \sum_{k=-\infty}^{\infty} h(k) \right\rvert < \infty$	$\int_{-\infty}^{\infty} h(\tau) d\tau < \infty$
Invertible	$h[n] * h^{-1}[n] = \delta[n]$	$h(t) * h^{-1}(t) = \delta(t)$

Example 5:

$$h(t) = e^{-3t} u(t-1)$$



Memoryless: $h \neq 0 \; ; \; t \neq 0$

the system has memory

Causal: $h(t) = 0 \; ; \; t < 0$

the system is causal.

Stability

$$\int_{-\infty}^{\infty} h(\tau)\, d\tau$$

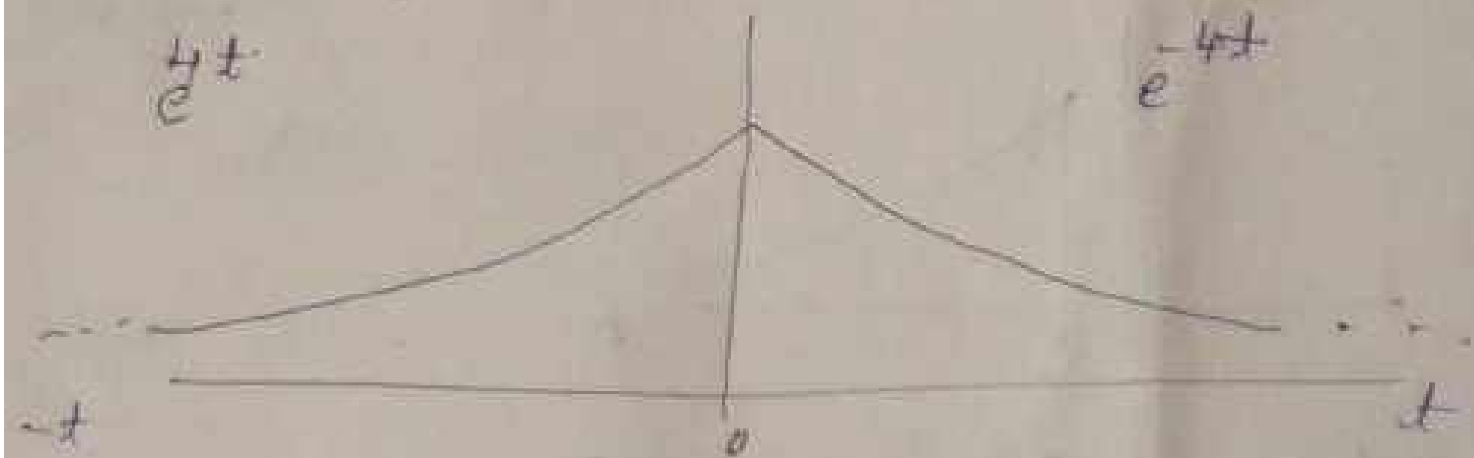
$$= \int_{1}^{\infty} e^{-3t}\, dt = \left.\frac{e^{-3t}}{-3}\right]_{1}^{\infty}$$

$$= \frac{e^{-3}}{3} < \infty$$

the system is stable

Example 6.

$$h(t) = e^{-4|t|}$$



Memoryless: $h(t) \neq 0 \quad t \neq 0$

It has memory.

Causal $h(t) \neq 0 \quad t < 0$

the system is non causal.

Stability

$$\int_{-\infty}^{\infty} e^{-4|t|}\,dt$$

$$= \int_{-\infty}^{0} e^{4t}\,dt + \int_{0}^{\infty} e^{-4t}\,dt.$$

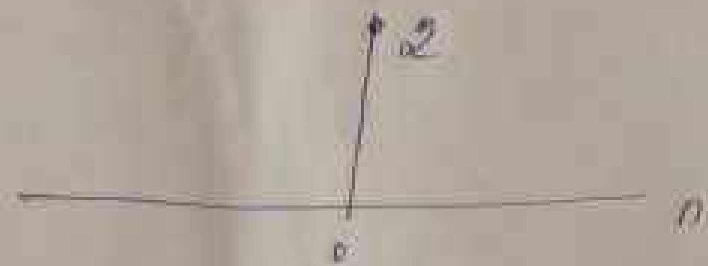
$$= \left[\frac{e^{4t}}{4}\right]_{-\infty}^{0} + \left[\frac{e^{-4t}}{-4}\right]_{0}^{\infty}$$

$$= \frac{1}{2} < \infty$$

the system is stable

Example 7:

$$h[n] = 2u[n] - 2u[n-1]$$



Memoryless $\quad h[n] = 0 \quad n \neq 0$

the system is memoryless

Causal $\quad h[n] = 0 \; ; \; n<0$

the system is causal.

Stability $\quad 2 < \infty$

the system is stable

Assignment

$$h[n] = \delta[n] + \sin n\pi$$

$$h(t) = 3\,\delta(t)$$

$$h[n] = 4^{-n}\,u[2-n]$$

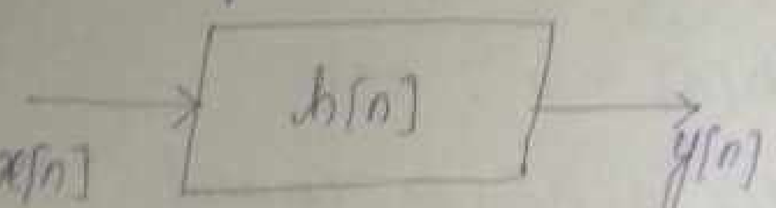
$$h[n] = n\left(\frac{1}{2}\right)^{n} u[n]$$

$$h(t) = e^{t}\,u(-t-1)$$

Step response of LTI system

The response of a LTI system to a step characterizes how the system responds to sudden changes in the input.

The step response is expressed in terms of the impulse response using convolution.



$$y[n] = x[n] * h[n]$$

if $x[n] = u[n]$ then $y[n] = s[n]$

Thus $\boxed{s[n] = u[n] * h[n]}$

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]$$

since $u[n-k] = 0$ for $k > n$

and

$u[n-k] = 1$ for $k \leq n$, we have

$$\boxed{s[n] = \sum_{k=-\infty}^{n} h[k]}$$

i.e., the step response is the running sum of the impulse response.

Similarly, the step response of continous time system is expressed as the running integral of the impulse response

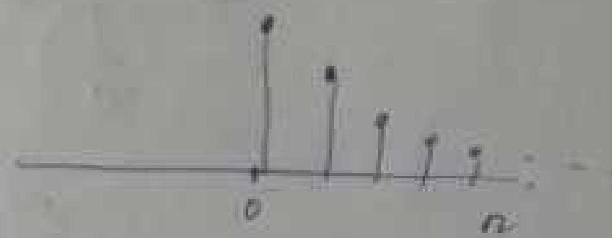
$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau}$$

The impulse response is expressed in terms of the step response as

$$\boxed{h(t) = \frac{d}{dt} s(t)}$$

Find the step response of LTI system for the impulse responses given

(i) $h[n] = \left(\frac{1}{2}\right)^n u[n]$



$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^n$$

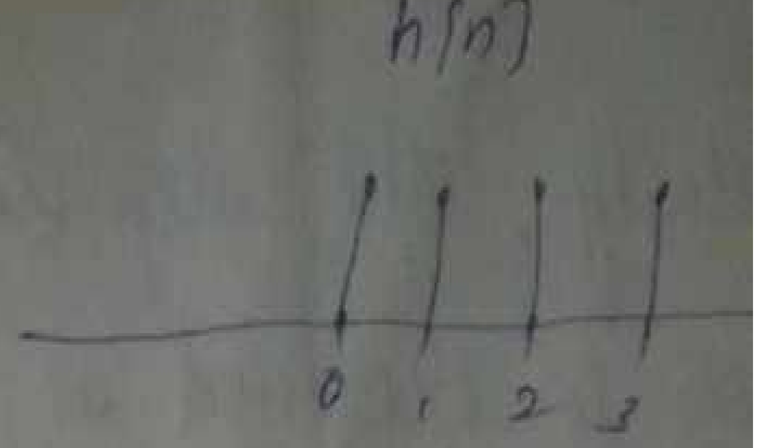
$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

$$s[n] = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

$$s[n] = 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right]$$

{2}

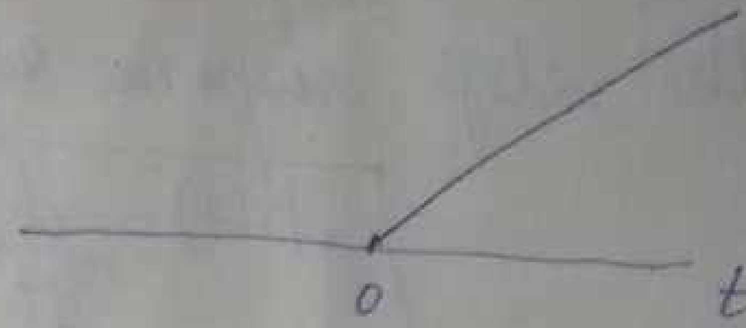
$$h[n] = u[n]$$



$$s[n] = \sum_{k=0}^{n} 1$$

$$s[n] = n+1$$

⟨3⟩ . $h(t) = t \cdot u(t)$



$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$= \int_{0}^{t} \tau \cdot d\tau = \left.\frac{\tau^2}{2}\right]_0^t$$

$$s(t) = \frac{t^2}{2}$$

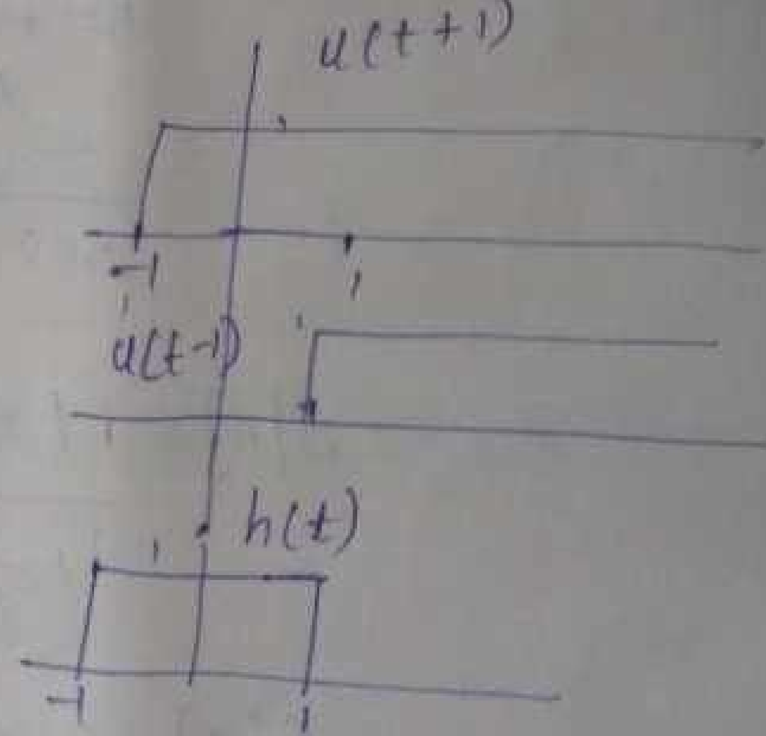
(4) $h(t) = u(t+1) - u(t-1)$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$s(t) = \int_{-1}^{1} 1 \cdot d\tau$$

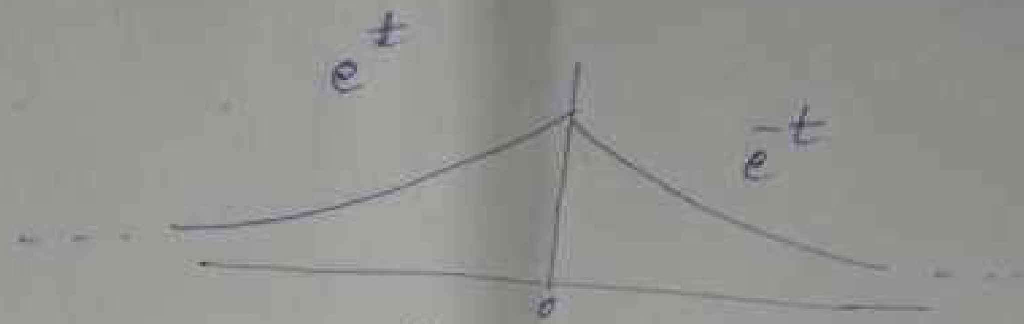
$$= \tau \Big]_{-1}^{1}$$

$$s(t) = 2$$



5)

$$h(t) = e^{-|t|}$$



$$s(t) = \int_{-\infty}^{t} e^{-|\tau|}\, d\tau$$

$$= \int_{-\infty}^{0} e^{\tau}\, d\tau + \int_{0}^{t} e^{-\tau}\, d\tau$$

$$= \left. e^{\tau} \right]_{-\infty}^{0} + \left[-e^{-\tau} \right]_{0}^{t}$$

$$= 1 + \left(-e^{-t} + e^{0} \right)$$

$$s(t) = 2 - e^{-t}$$

Assignment.

<i> $h[n] = (-a)^n u[n]$

<ii> $h[n] = e^{-2|n|}$

<iii> $h(t) = e^{2t} u(t-1)$

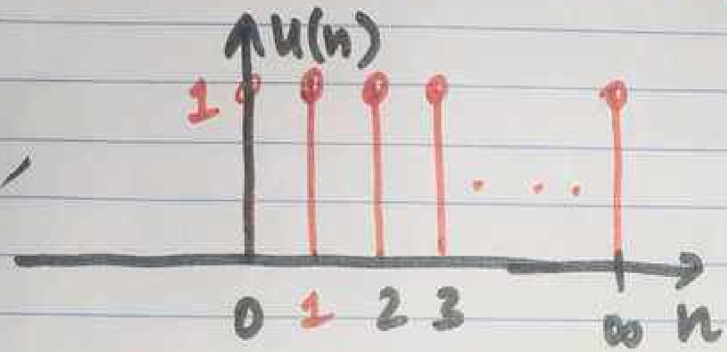
<iv> $h(t) = u(t)$

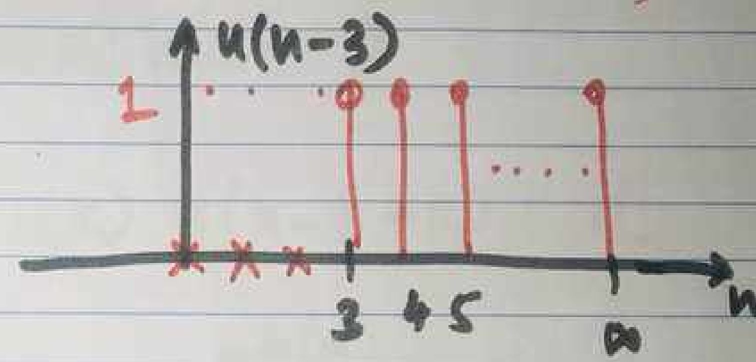
Convolution Sum

RSPL

2 INFINITE SEQUENCES

EXAMPLE 4 : $x(n) = u(n)$ $h(n) = u(n-3)$





$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & ow \end{cases} \qquad u(n-3) = \begin{cases} 1 & n \geq 3 \\ 0 & ow \end{cases}$$

$\therefore$ Convolution Sum has 2 cases

1. case 1 $n < 3$ or $(n-3<0)$
2. case 2 $n \geq 3$ or $(n-3 \geq 0)$

WKT $y[n] = x(n) * h(n)$

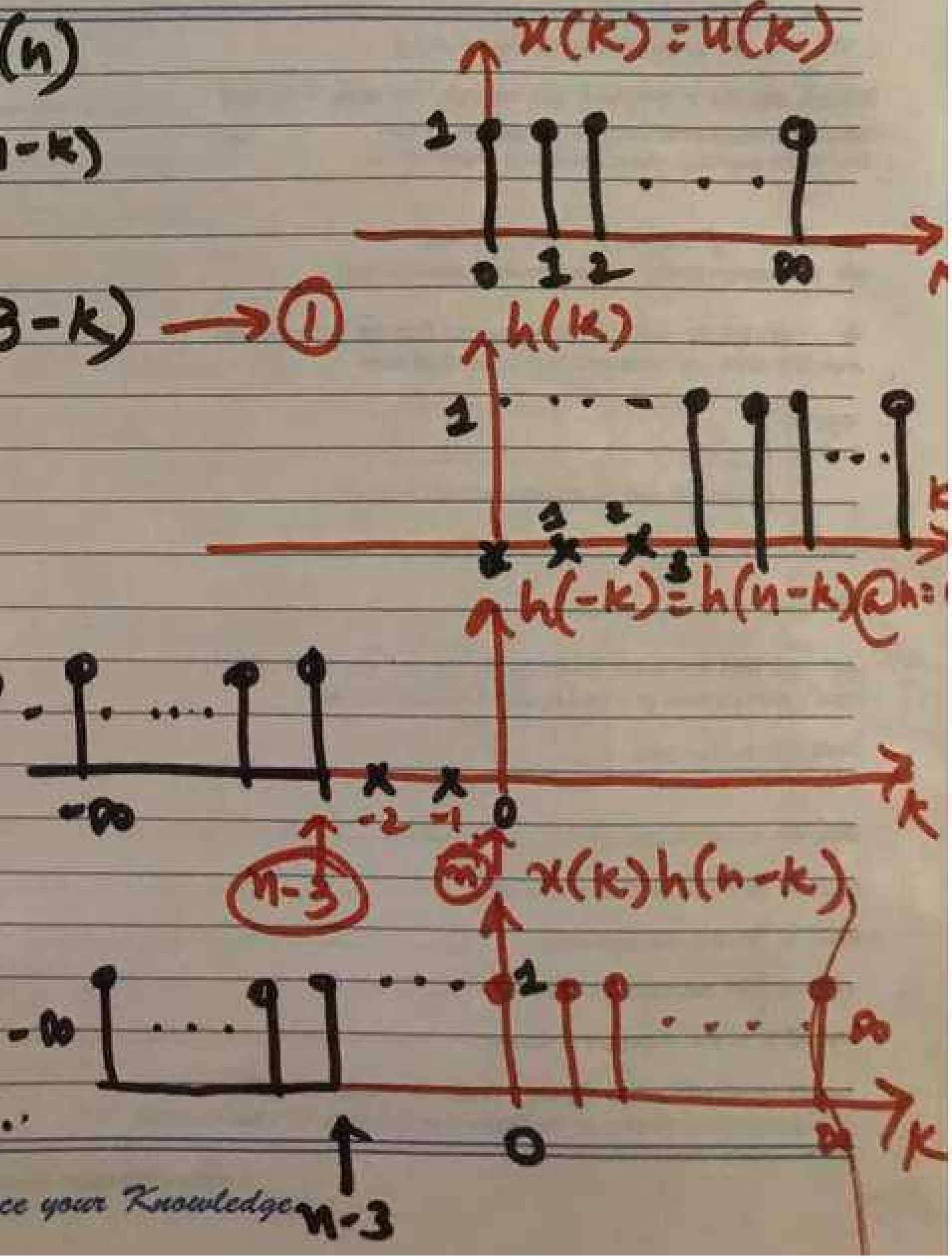
$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} u(k)u(n-3-k) \longrightarrow (1)$$

Case 1: $n<3$ or $(n-3)<0$

$y[n]=0$

As there is no overlapping b/w the samples of $x(k)$ & $h(n-k)$

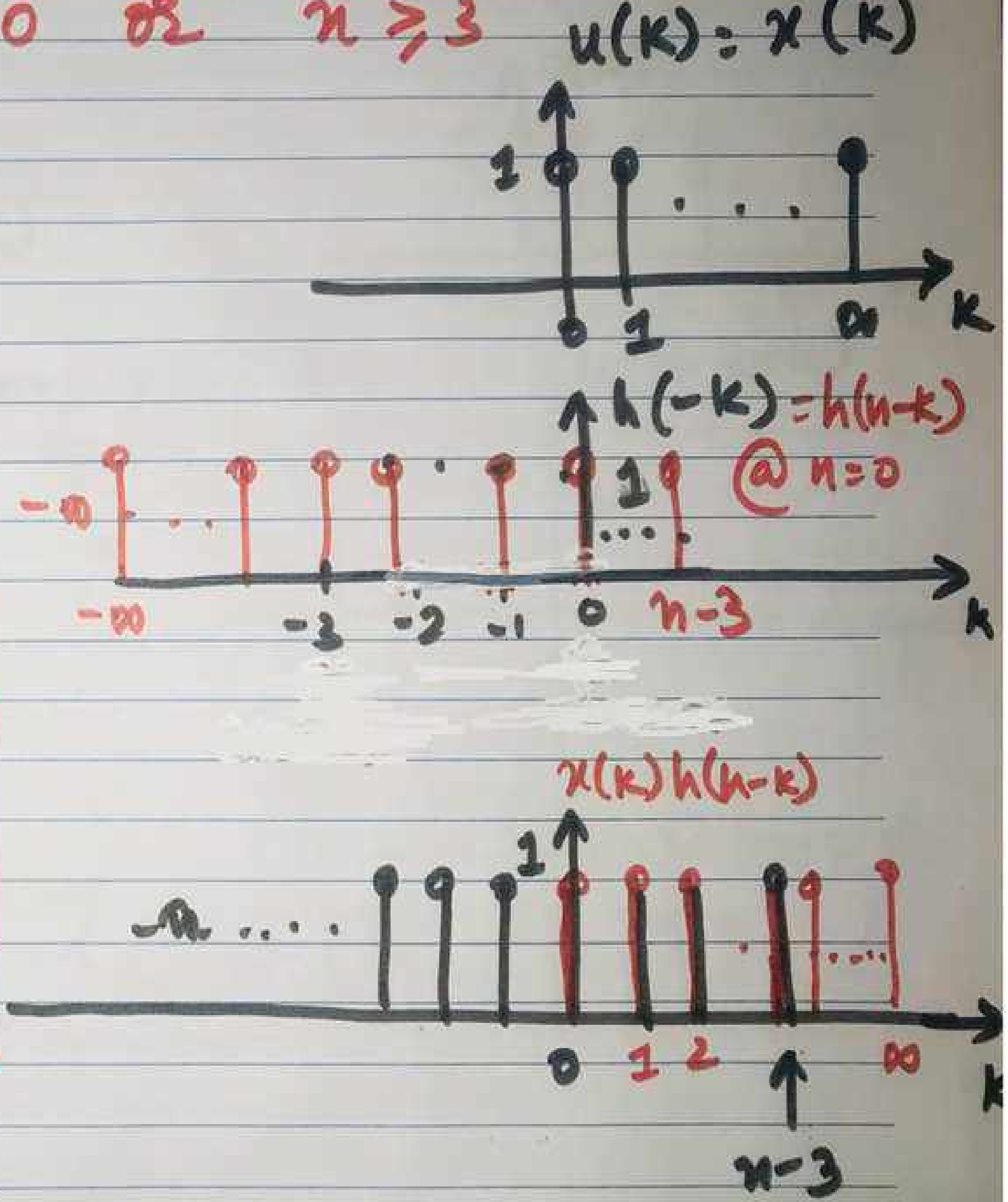


Case 2 : $n-3 \geq 0$ or $n \geq 3$

$$① \Rightarrow y(n) = \sum_{k=0}^{n-3} 1$$

$$= n-3+1$$

$$y(n) = n-2$$



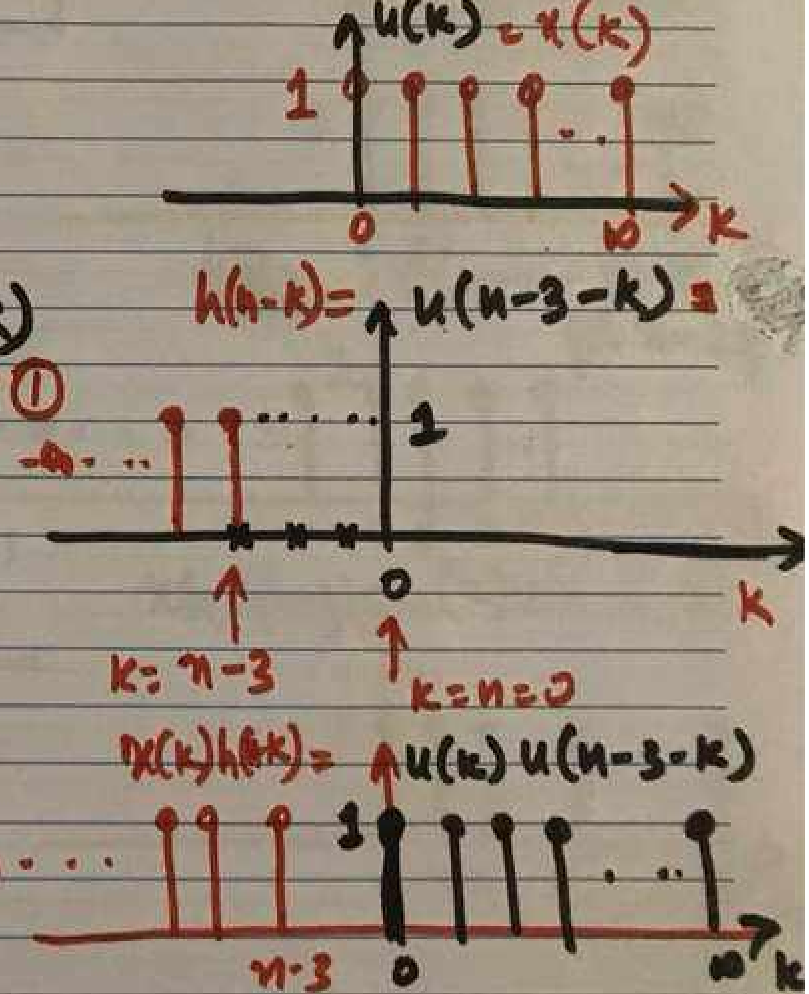
$$\therefore \boxed{y(n) = \begin{cases} 0 & n < 3 \\ n-2 & n \geq 3 \end{cases}}$$

EXAMPLE 5: $x(n) = \beta^n u(n)$ $\quad h(n) = u(n-3)$

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \beta^k u(k)\, u(n-3-k) \longrightarrow (1)$$



$\boxed{\text{Case 1 : } n-3 < 0 \ \& \ n < 3}$

(1) $\Rightarrow$ $y[n] = \sum_{k=-\infty}^{\infty} \beta^k \cdot 0$

$y[n] = 0$

As there is no overlapping b/w the samples of $x(k)$ & $h(n-k)$

$\boxed{\text{Case 2: } n-3 \geq 0 \ \& \ n \geq 3}$

$$y[n] = \sum_{k=0}^{n-3} \beta^k \cdot 1.$$

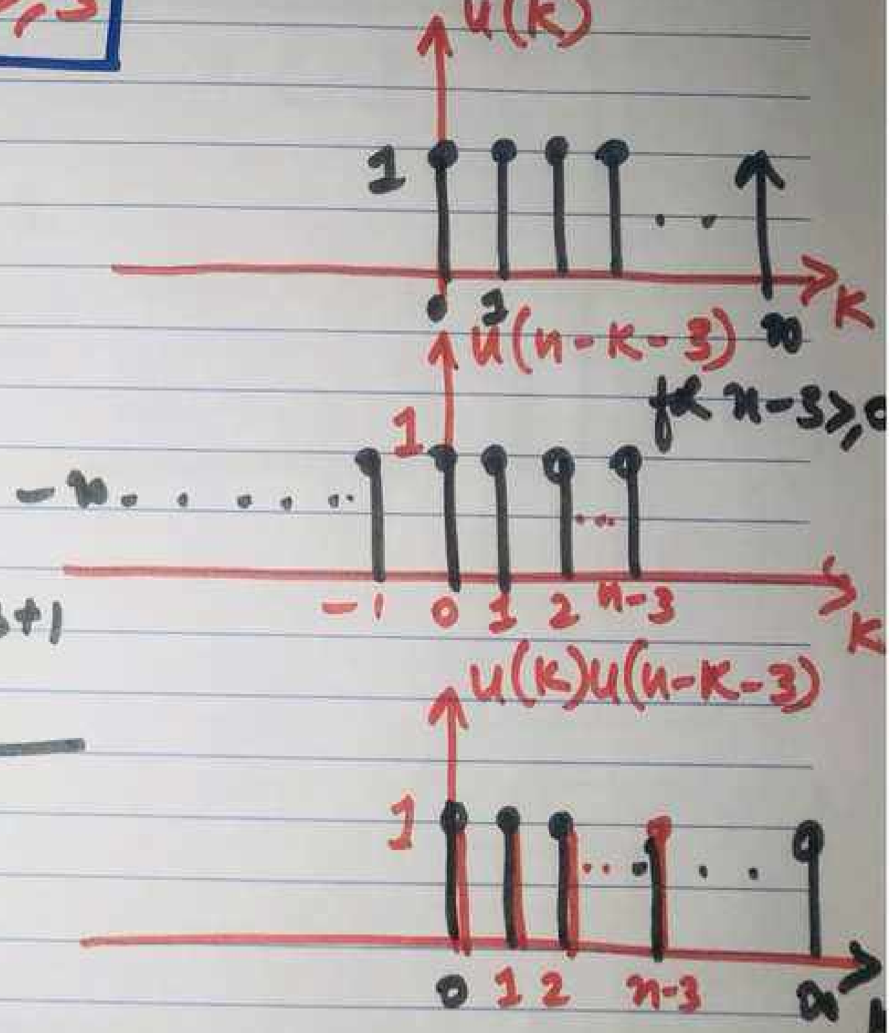
w.k.t

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

IIIly

$$y[n] = \sum_{k=0}^{n-3} \beta^k = \frac{1-\beta^{n-3+1}}{1-\beta}$$

$$y[n] = \frac{1-\beta^{n-2}}{1-\beta}$$



$$\therefore \boxed{y[n] = \begin{cases} 0 & n < 3 \\ \dfrac{1-\beta^{n-2}}{1-\beta} & n \geq 3 \end{cases}}$$

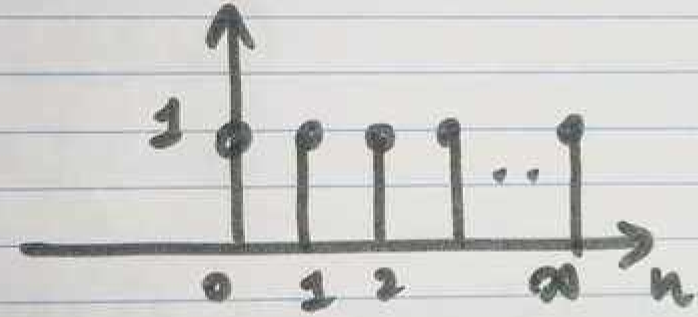
RSPL

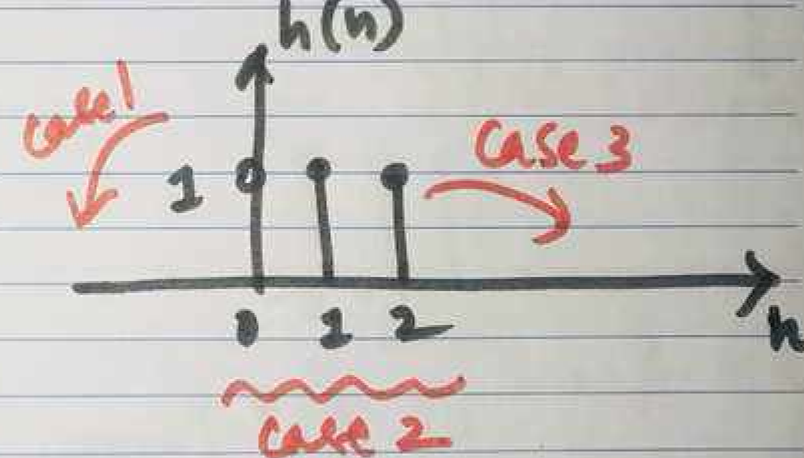
1 finite & 1 Infinite Sequence

Example 7 : $x(n) = u(n)$ $\quad$ $h(n) = u(n) - u(n-3)$

In case of 1 finite & 1 Infinite case there will be 3 cases.

$x(n) = u(n)$





Case 1 : $n < 0$

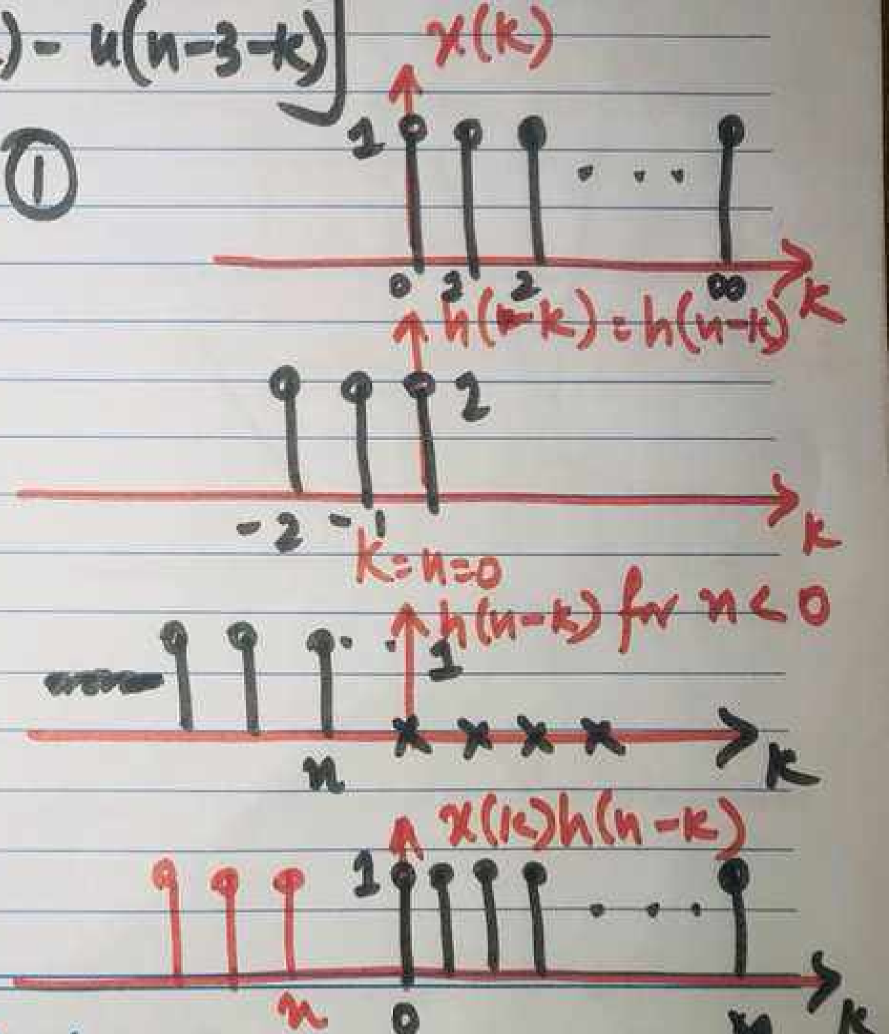
case 2 : $0 < n \leq 2$

case 3 : $n \geq 2$

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} u(k)\left[u(n-k) - u(n-3-k)\right] \longrightarrow ①$$



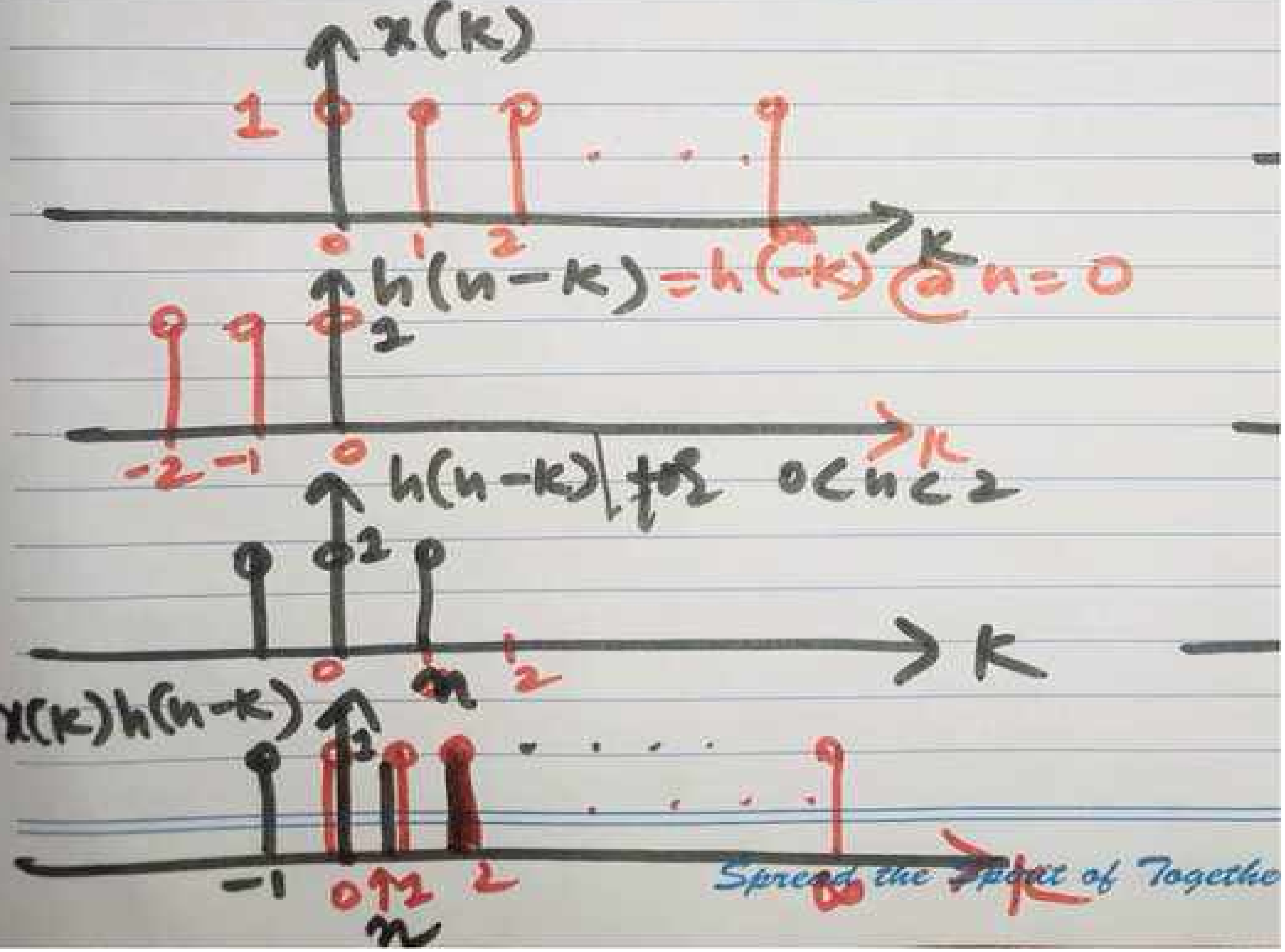
Case 1 : $n < 0$

① $y[n] = 0$

As there is no overlapping b/w the samples of $x(k)$ & $h(n-k)$

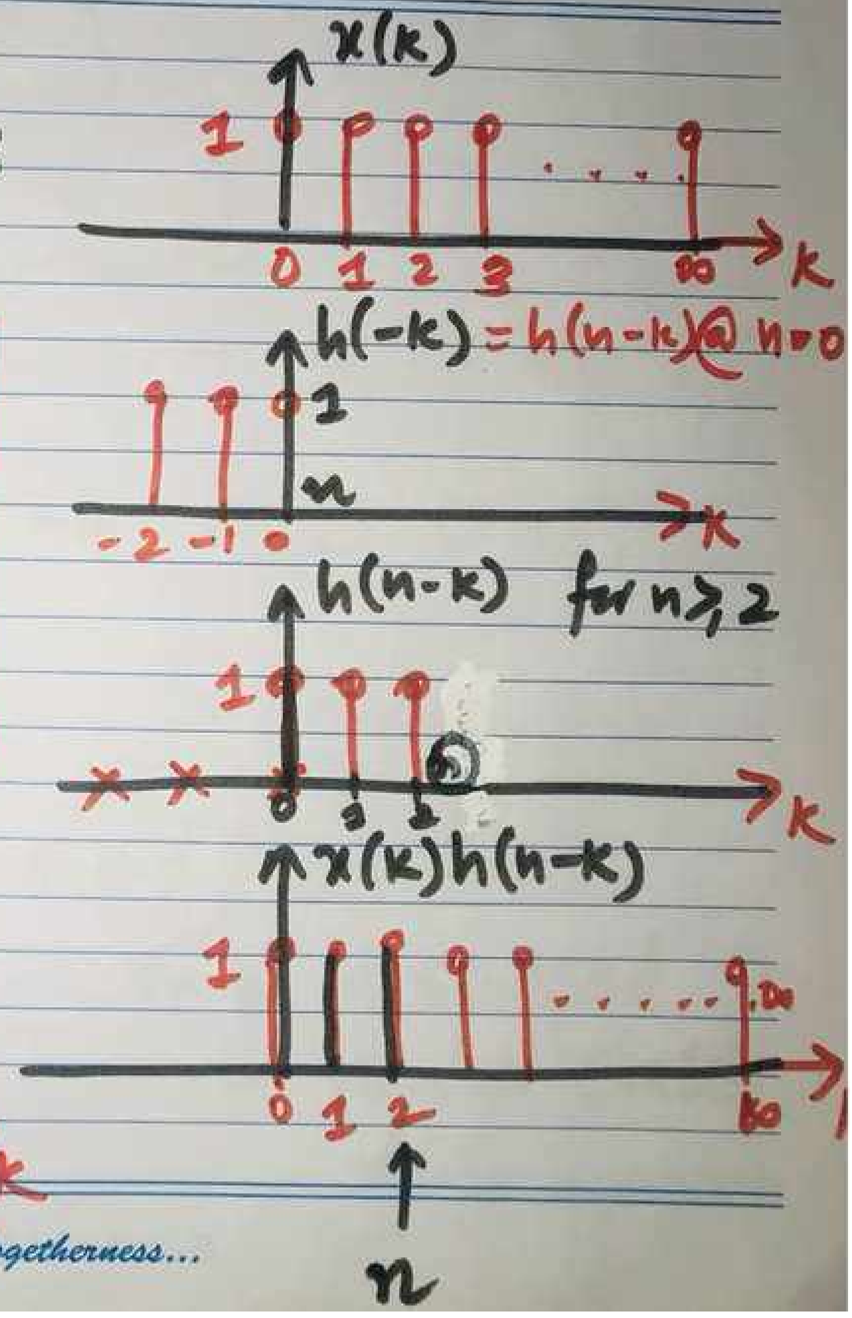
Case 2 :. $0 < n < 2$

$$y[n] = \sum_{k=0}^{n} 1. = n+1$$

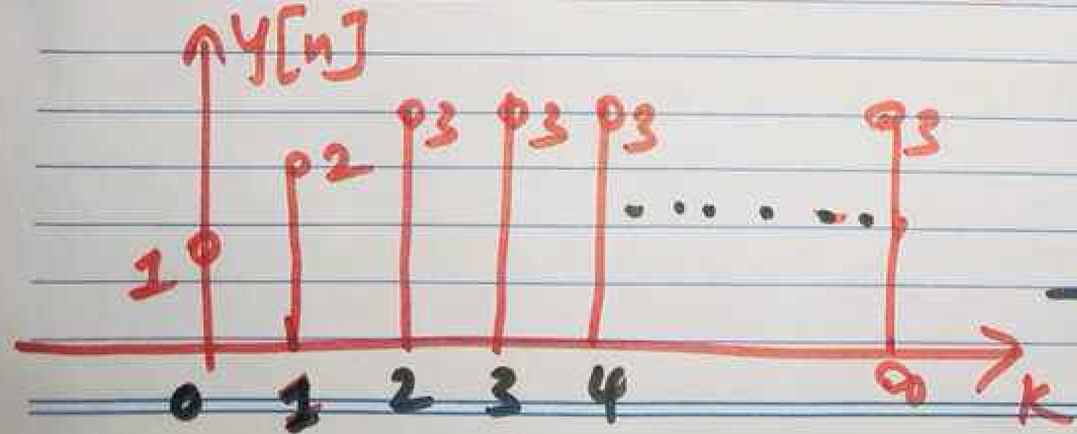


Case 3: $n \geq 2$

$$y[n] = \sum_{k=0}^{2} 1 = 1+1+1 = 3$$



$$\boxed{\therefore\ y[n] = \begin{cases} 0 & n < 0 \\ n+1 & 0 < n < 2 \\ 3 & n \geq 2 \end{cases}}$$



Both the signals are of finite duration

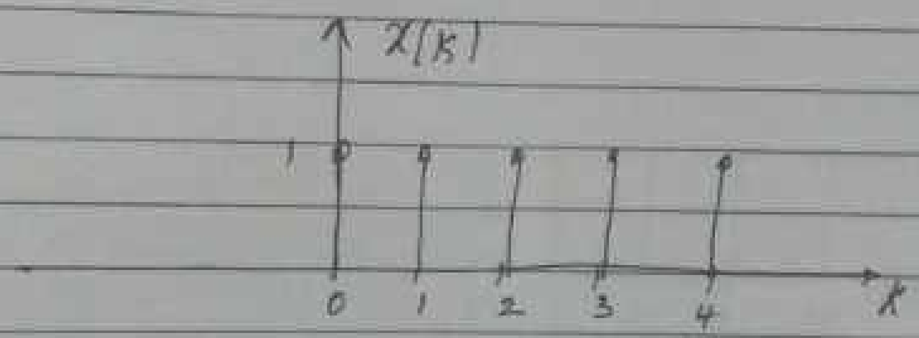
Example : 6

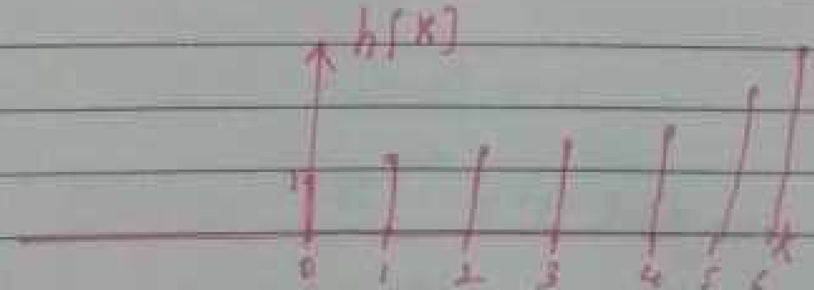
$$x(n) = 1 \quad ; \quad 0 \leq n \leq 4$$
$$= 0 \quad ; \quad \text{otherwise}$$

$$h(n) = \alpha^n \quad ; \quad 0 \leq n \leq 6$$
$$= 0 \quad ; \quad \text{otherwise} \qquad \alpha > 1$$

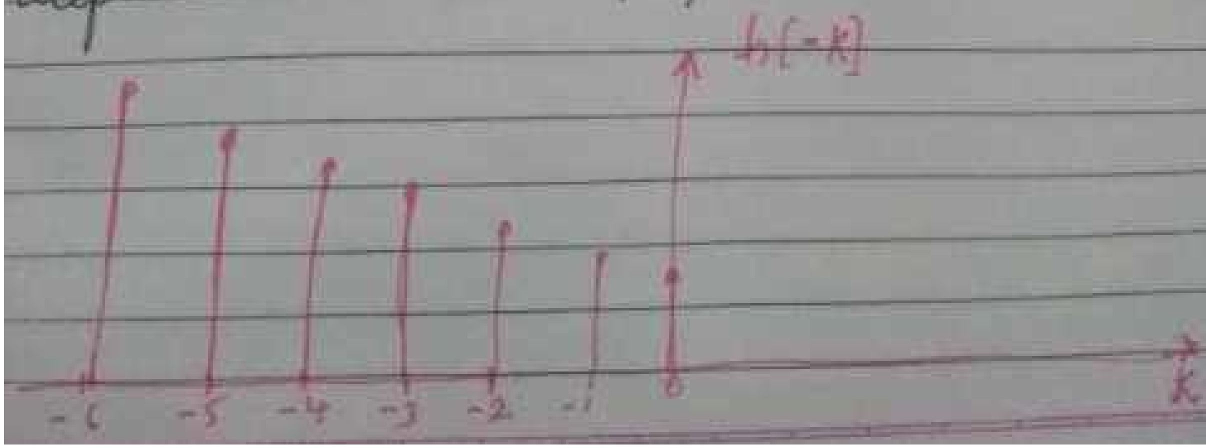
$$y(n) = x(n) * h(n)$$

Step 1: Sketch $x(k)$ & $h(k)$

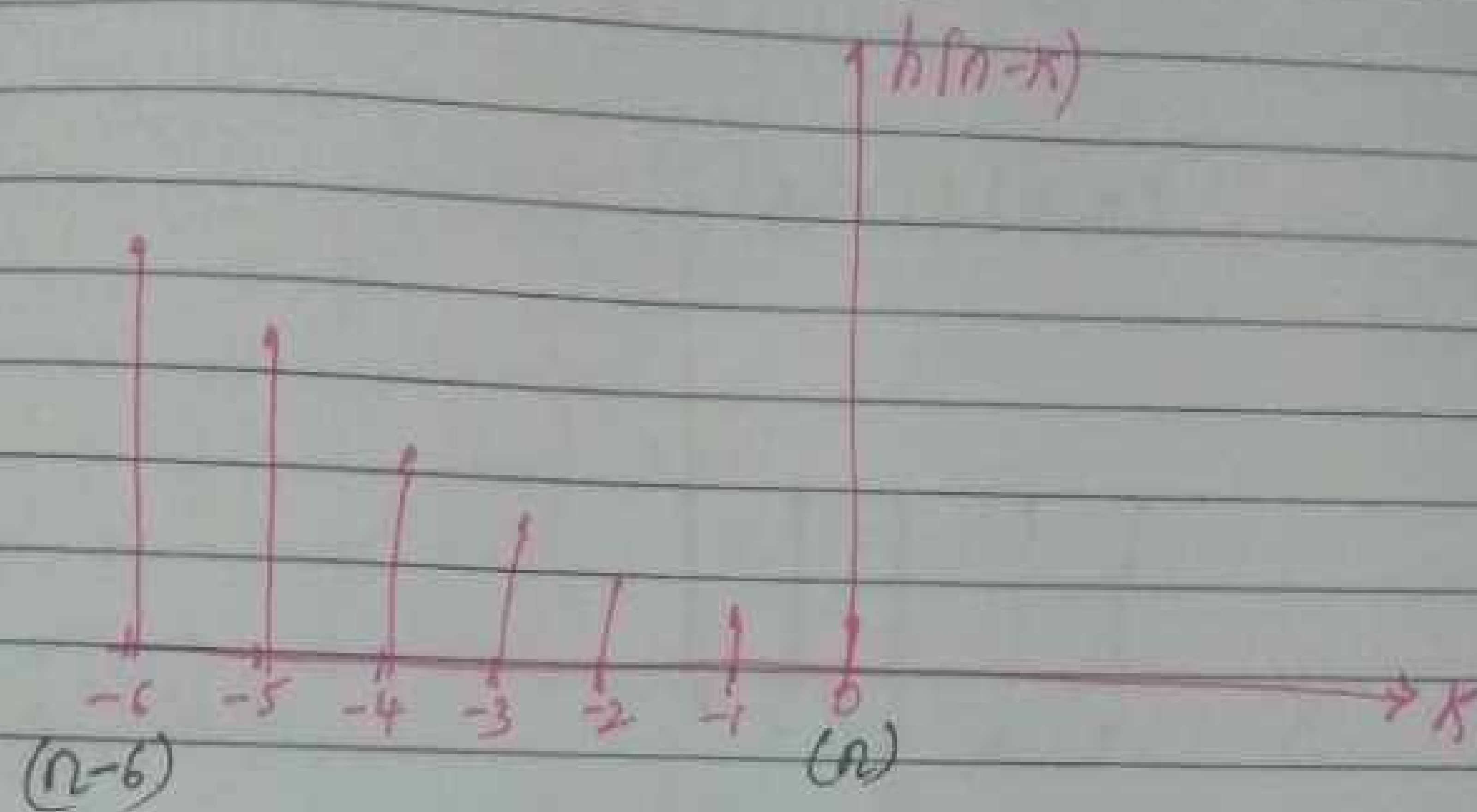




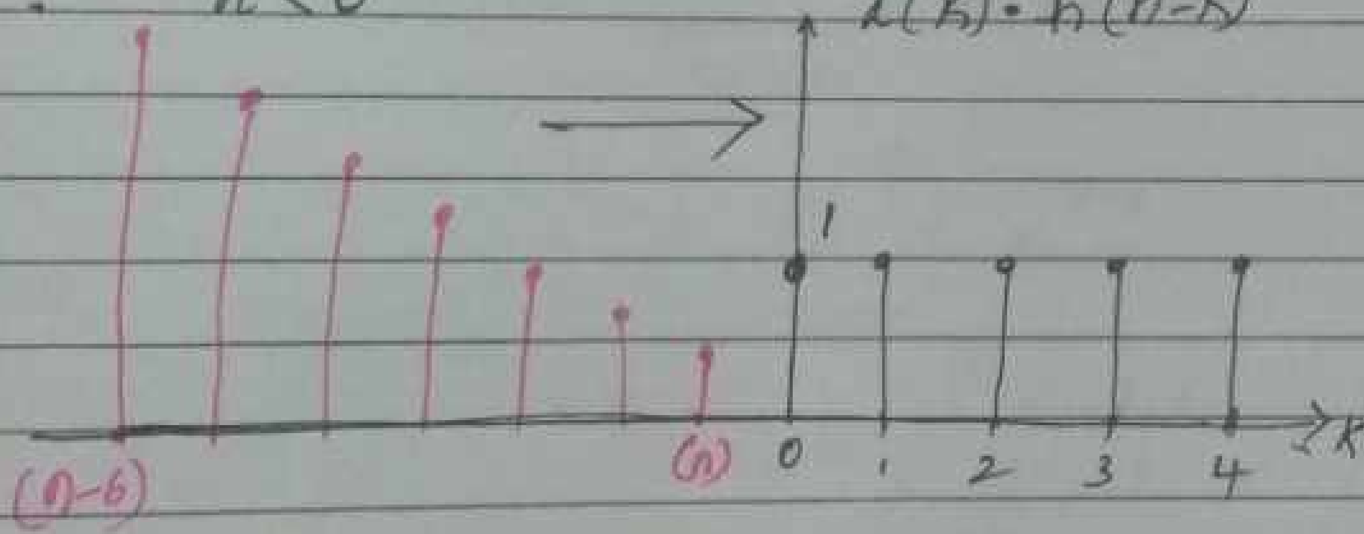
Step 2: Sketch $h[-k]$



Step 3 : Sketch $h[n-k]$



Case 1 : $n < 0$



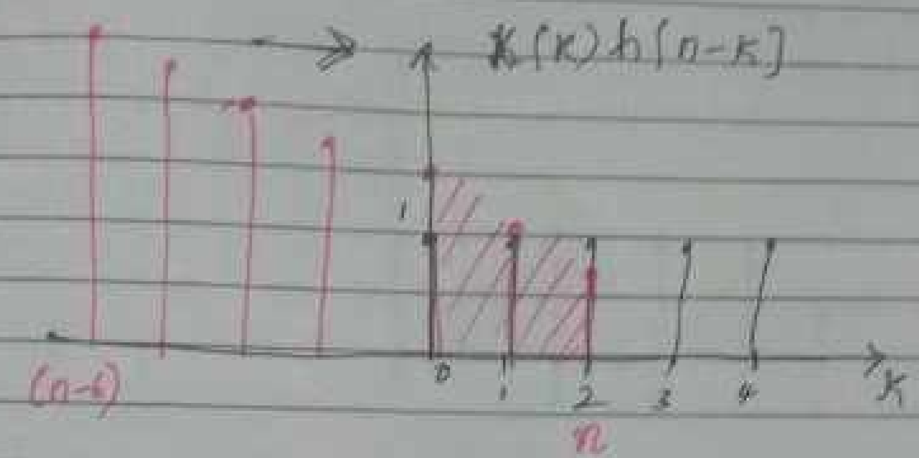
$$x(k)\, h(n-k) = 0$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$y(n) = 0$$

Case 2 : $n \geq 0$ & $n \leq 4$

i.e. $0 \leq n \leq 4$



$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$= \sum_{k=0}^{n} 1 \cdot \alpha^{n-k}$$

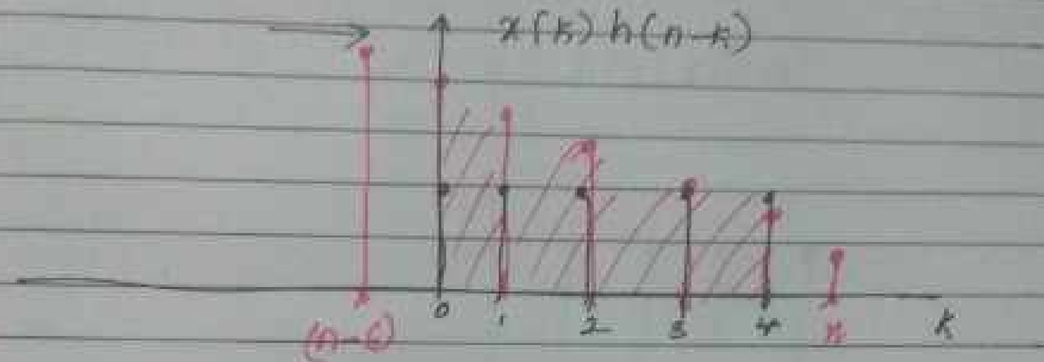
$$= \alpha^{n} \sum_{k=0}^{n} (\alpha^{-1})^{k}$$

$$\sum_{k=0}^{n} a^{k} = \frac{1-a^{n+1}}{1-a}$$

$$y(n) = \alpha^{n}\left[\frac{1-(\alpha^{-1})^{n+1}}{1-(\alpha^{-1})}\right]$$

Case 3 : $n > 4$ and $(n-6) \leq 0$

$4 < n \leq 6$



$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=0}^{4} 1 \cdot \alpha^{n-k}$$

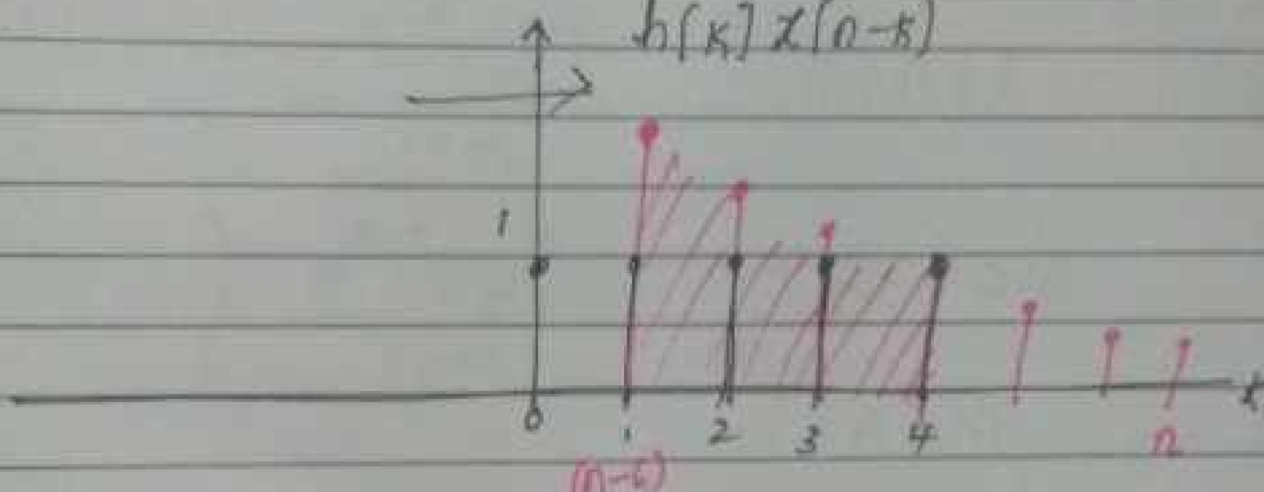
$$= \alpha^{n} \sum_{k=0}^{4} (\alpha^{-1})^{k}$$

$$= \alpha^{n} \left[\frac{1-(\alpha^{-1})^{5}}{1-(\alpha^{-1})} \right]$$

$$y(n) = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}$$

Case 4: $(n-6) \leq 4$ $(n-6) > 0$

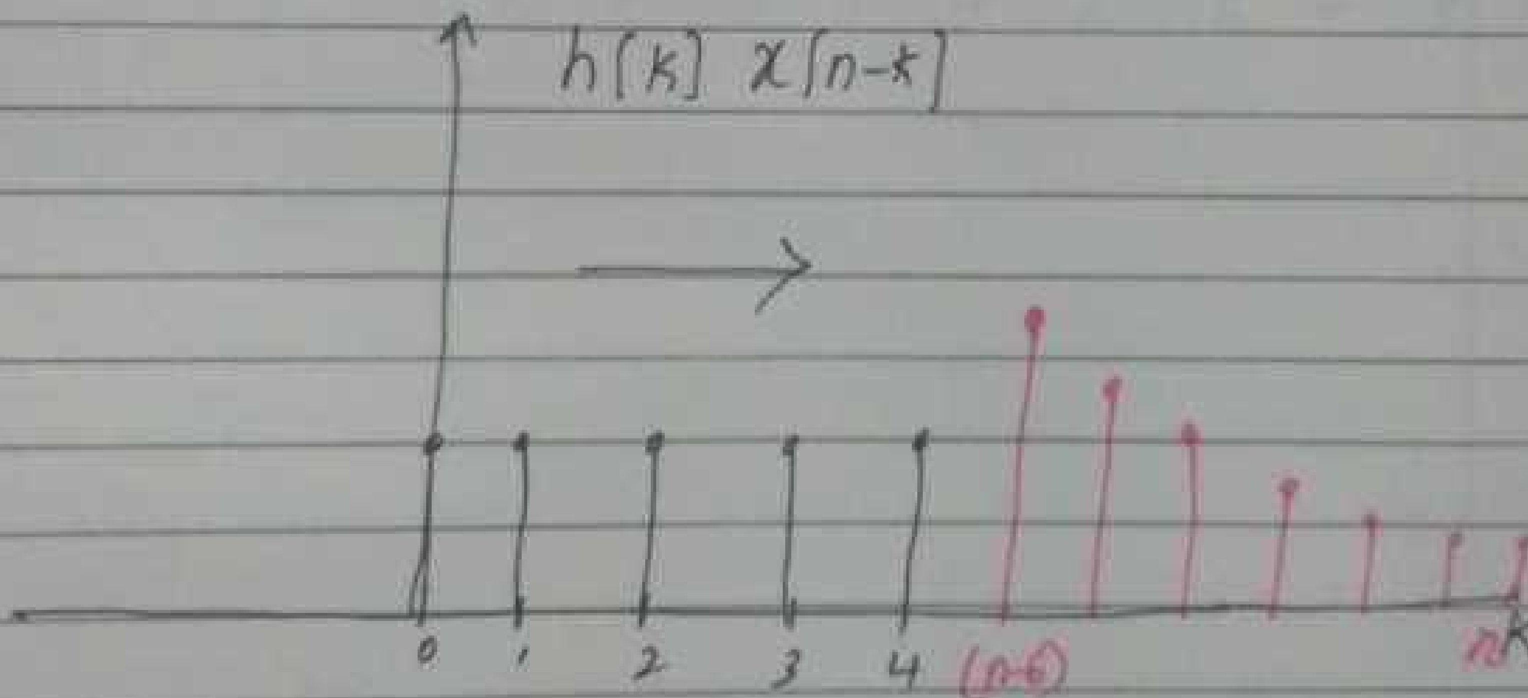
$6 < n \leq 10$



$$y[n] = \sum_{k=n-6}^{4} 1 \cdot \alpha^{n-k}$$

$$y[n] = \frac{\alpha^{n-4} - \alpha^{7}}{1-\alpha}$$

Case 5 : $(n-6) > 4$ i.e $n > 10$



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = 0$$

$$y(n) = \begin{cases} 0 & n<0 \\ \alpha^{n}\left(\dfrac{1-(\alpha^{-1})^{n+1}}{1-(\alpha^{-1})}\right) & 0\leq n \leq 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} & 4<n\leq 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha} & 6<n\leq 10 \\ 0 & n>10 \end{cases}$$

SUMMARY

No. of cases for y[n]

1. 2 infinite sequences — 2 cases
2. 1 infinite & 1 finite — 3 cases
3. 2 finite cases — 5 cases

   or

   (General Method)

ASSIGNMENT FOR TODAY

1) $x(n) = u(n)$ $\quad h(n) = u(n-5)$

2) $x(n) = \left(\frac{1}{2}\right)^n u(n)$ $\quad h(n) = u(n-2)$

3) $h(n) = u(n) - u(n-10)$

$x(n) = u(n-2) - u(n-7)$

Convolution Integral

Part II

Infinite duration with finite duration

$y(t) = x(t) * h(t)$.

$x(t) = u(t)$

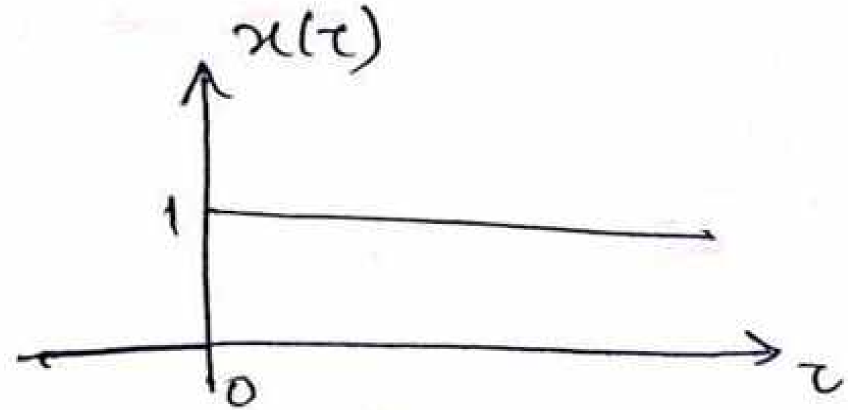
$$h(t) = \begin{cases} 1 & 1 \le t \le 3 \\ 0 & \text{Elsewhere} \end{cases}$$

Compute $y(t)$.?

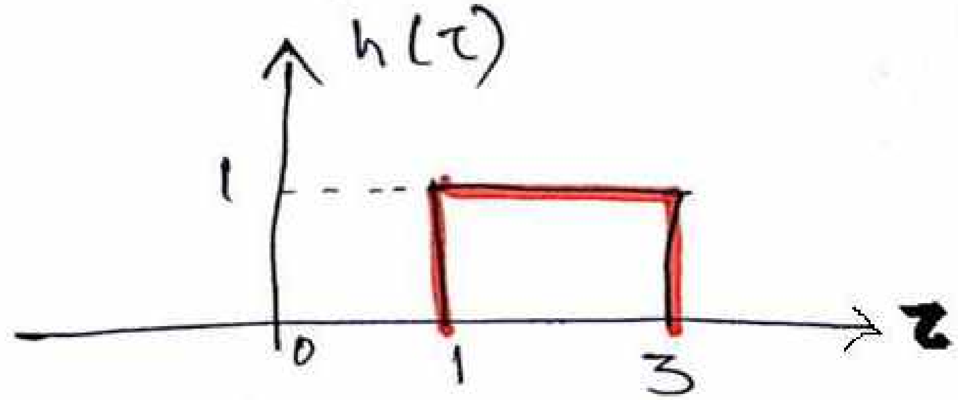
Solution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

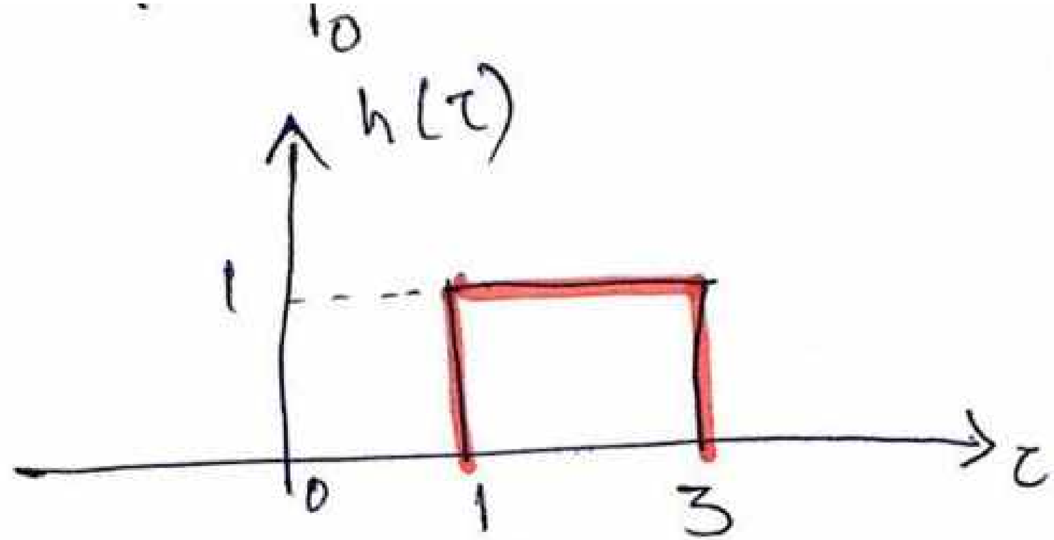
step 1 : Sketch $x(\tau)$



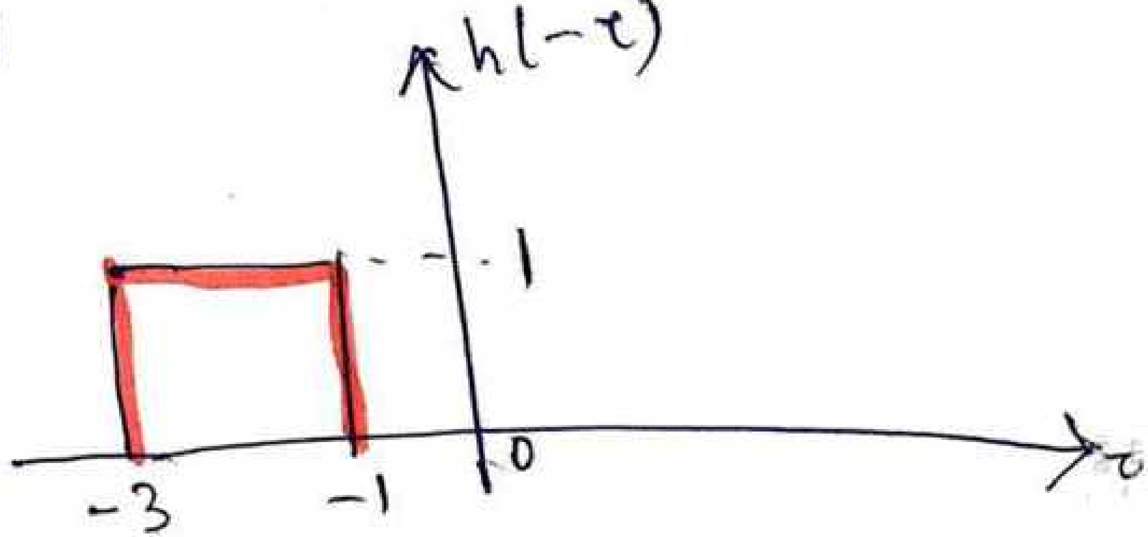
step 2 : Sketch $h(\tau)$



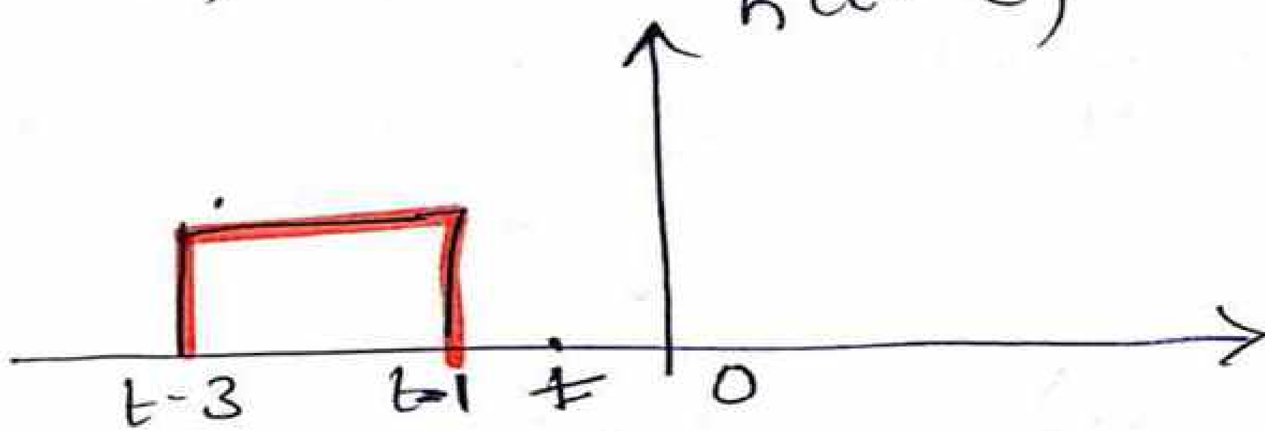
tep 2 : Sketch $h(\tau)$



ep 3 : Sketch $h(-\tau)$



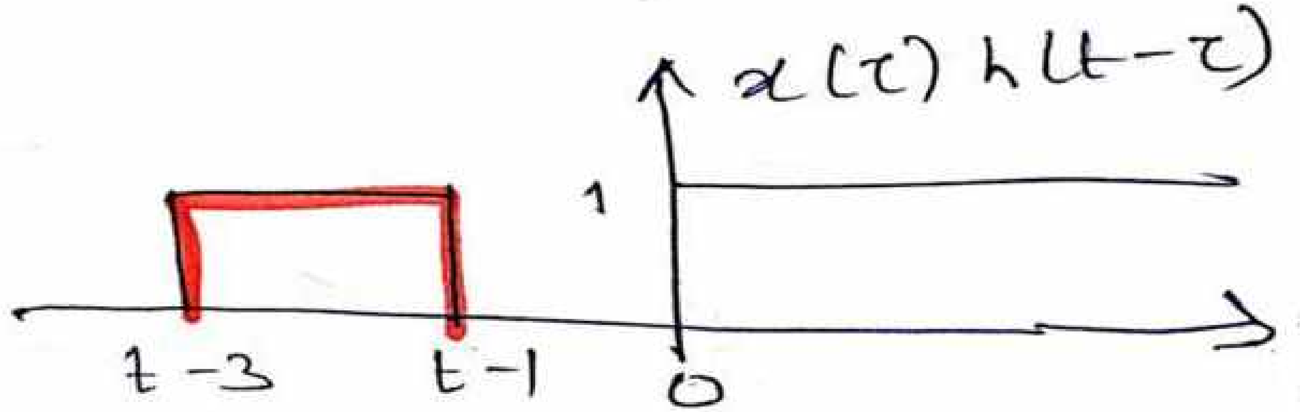
Step 4: Sketch $h(t-\tau)$



Case 1:

$t-1 \leq 0$

$t \leq 1$



$x(\tau)\, h(t-\tau) = 0$

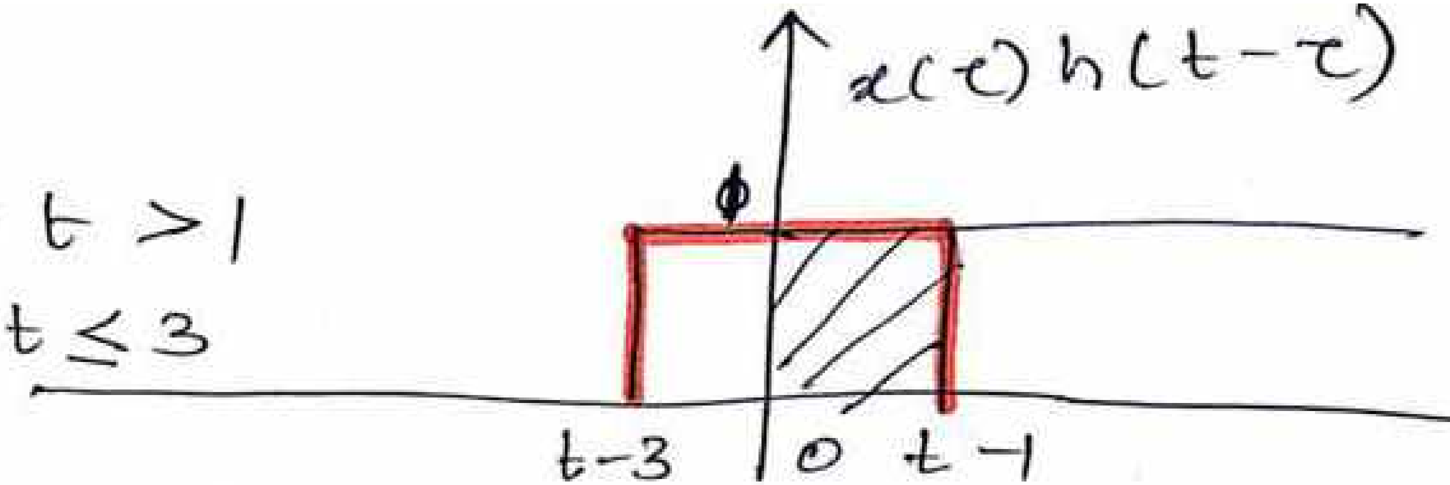
Product is zero.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \underline{0}$$

se 2:

$t-1 > 0$ (or) $t > 1$

$t-3 \le 0$ $\quad t \le 3$



$$(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

roduct existing between 0 to $t-1$

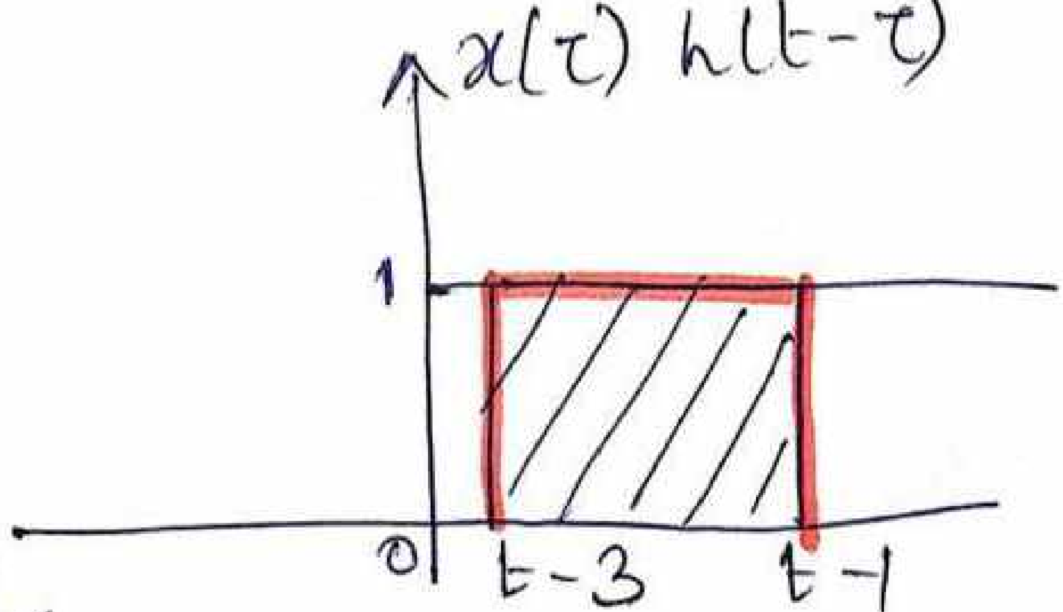
ence

$$y(t) = \int_0^{t-1} 1 \, d\tau = \tau \Big|_0^{t-1} = \underline{t-1}$$

__Case 3__ :

$t - 3 > 0$

$t > 3$



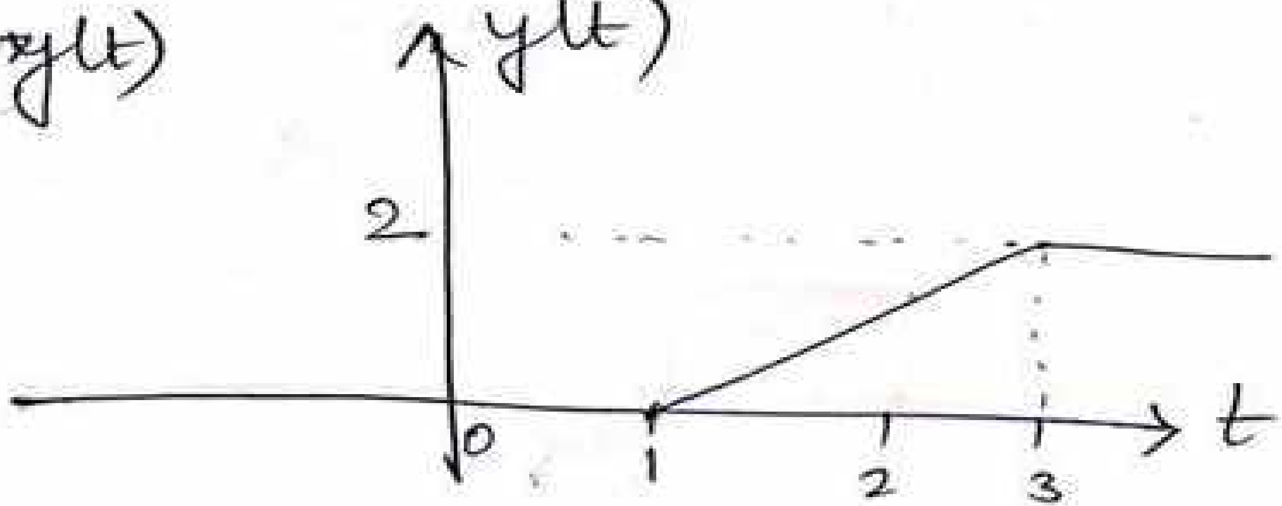
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{t-3}^{t-1} 1\,d\tau = \tau\Big|_{t-1}^{t-3} = (t-1) - (t-3) = \underline{\underline{2}}$$

Step 5

$$\therefore\ y(t) = \begin{cases} 0 & t \le 1 \\ t-1 & 1 < t \le 3 \\ 2 & t > 3 \end{cases}$$

Sketch ~~x~~ y(t)



Finite duration with finite duration

Example - 1

Compute $y(t)$ if

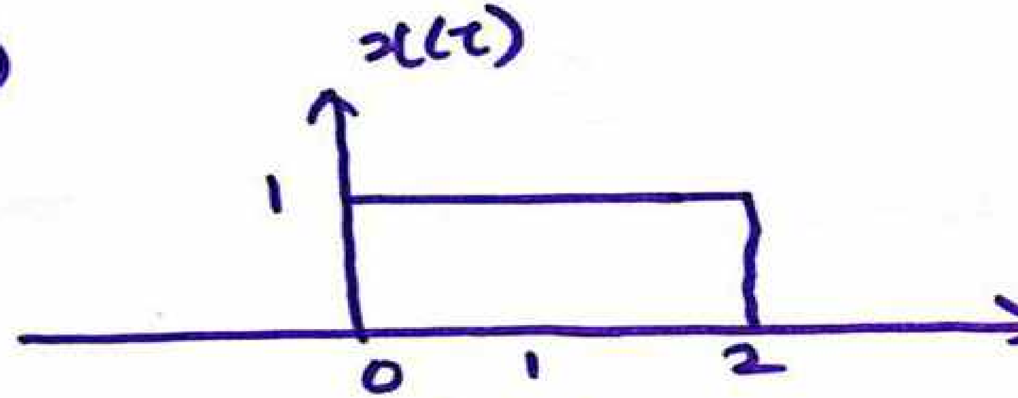
$$x(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

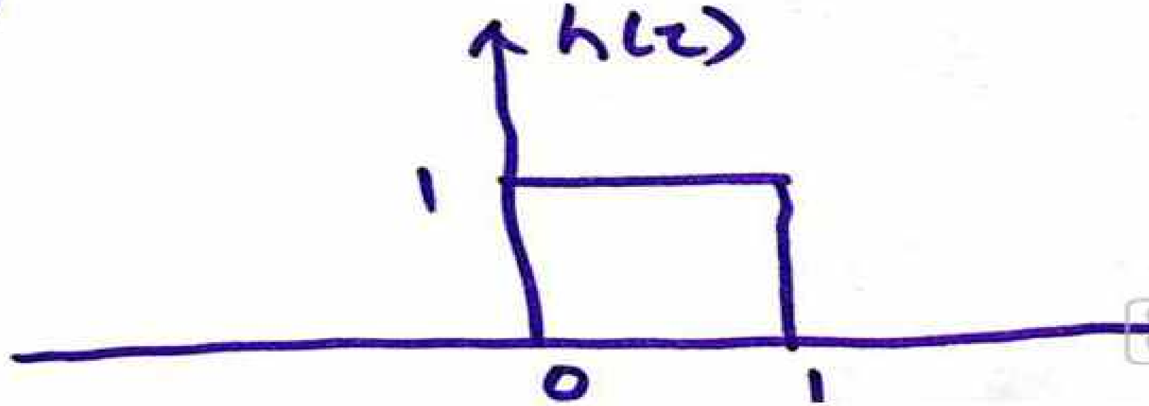
Soln.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

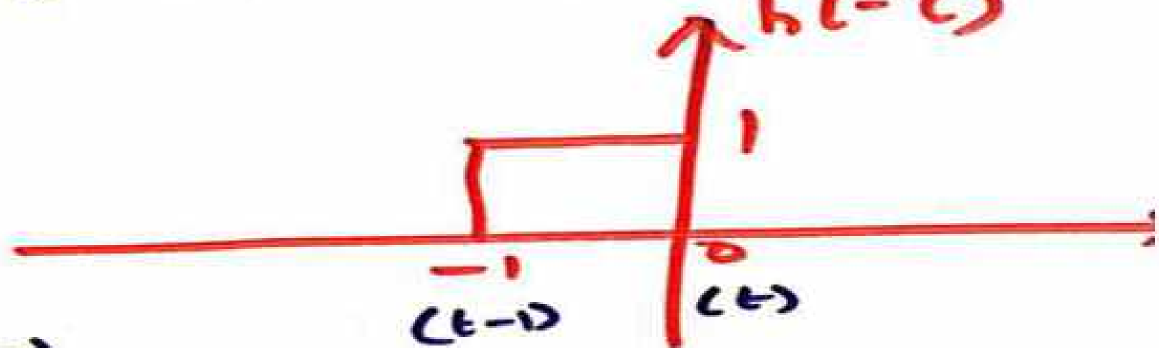
Step 1 - Sketch $x(\tau)$



Step 2 - Sketch $h(\tau)$

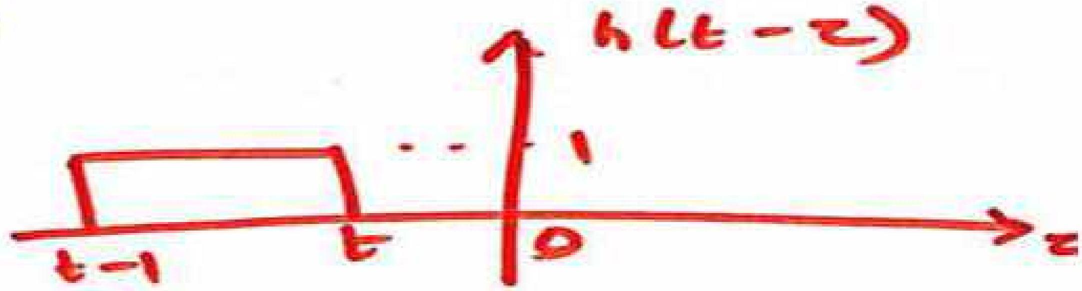


tep3 : Sketch $h(-\tau)$

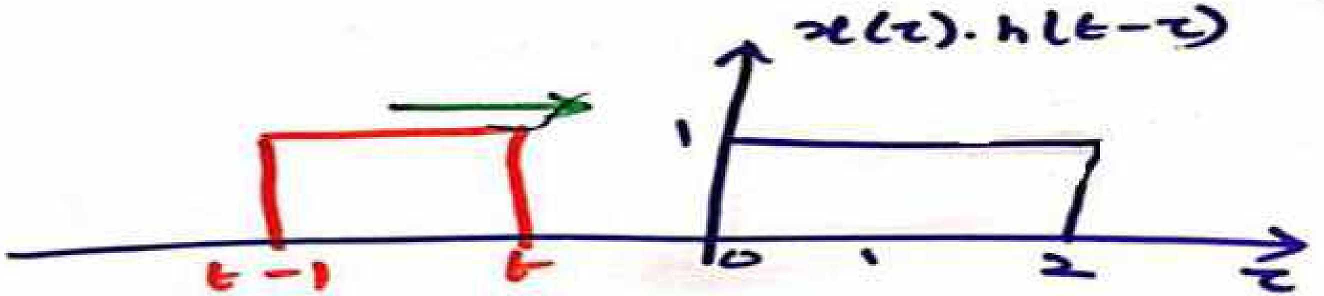


tep4 : Sketch $h(t-\tau)$

$= h(-(\tau - t))$



ase 1 : $t < 0$



$x(\tau) \cdot h(t-\tau) = 0$

hence

$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

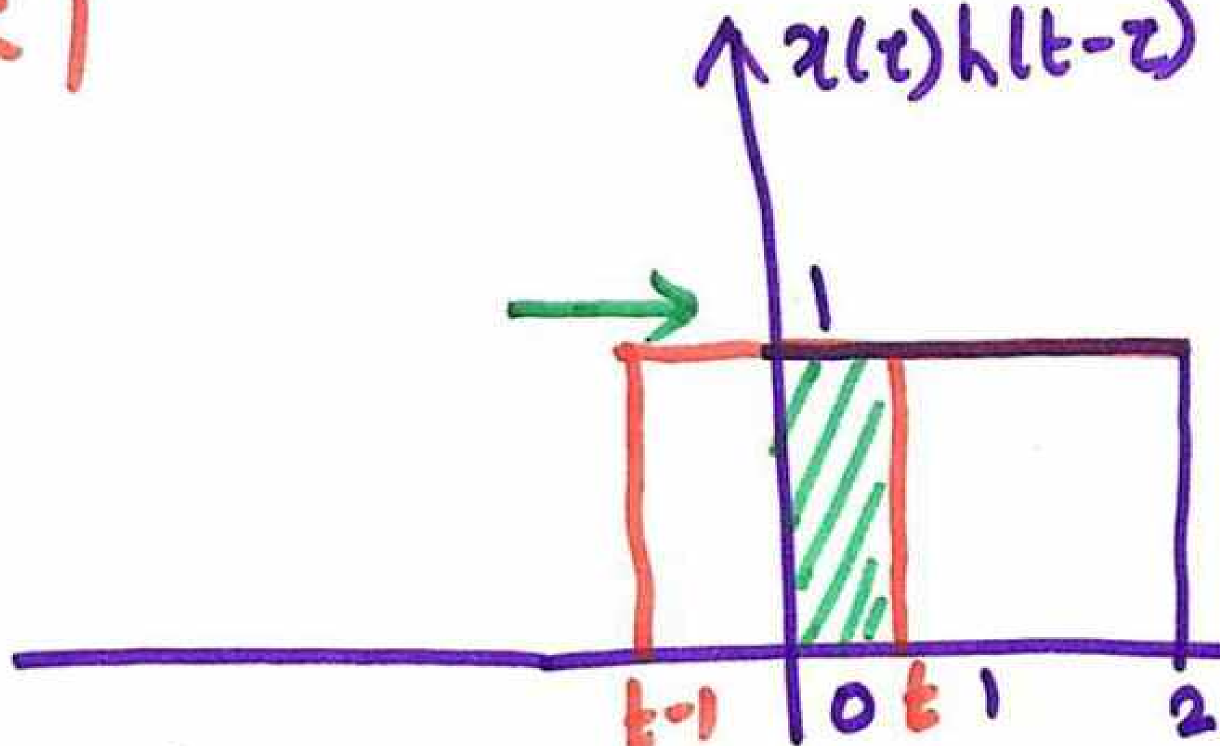
$= 0$

Case 2 : $0 < t < 1$

$$y(t) = \int_0^t 1 \, d\tau$$

$$= \tau \Big|_0^t$$

$$= t$$



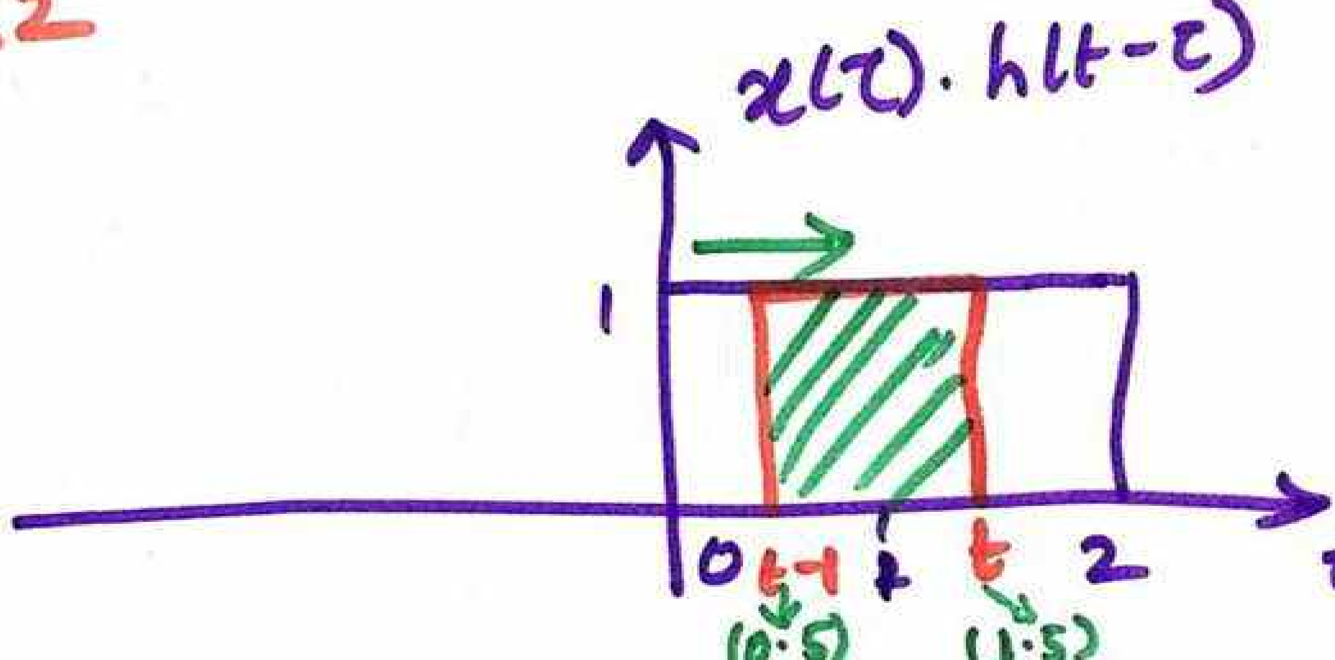
Case 3 : $1 < t < 2$

$$y(t) = \int_{t-1}^{t} 1 \, d\tau$$

$$= \tau \Big|_{t-1}^{t}$$

$$= t - t + 1$$

$$= 1$$

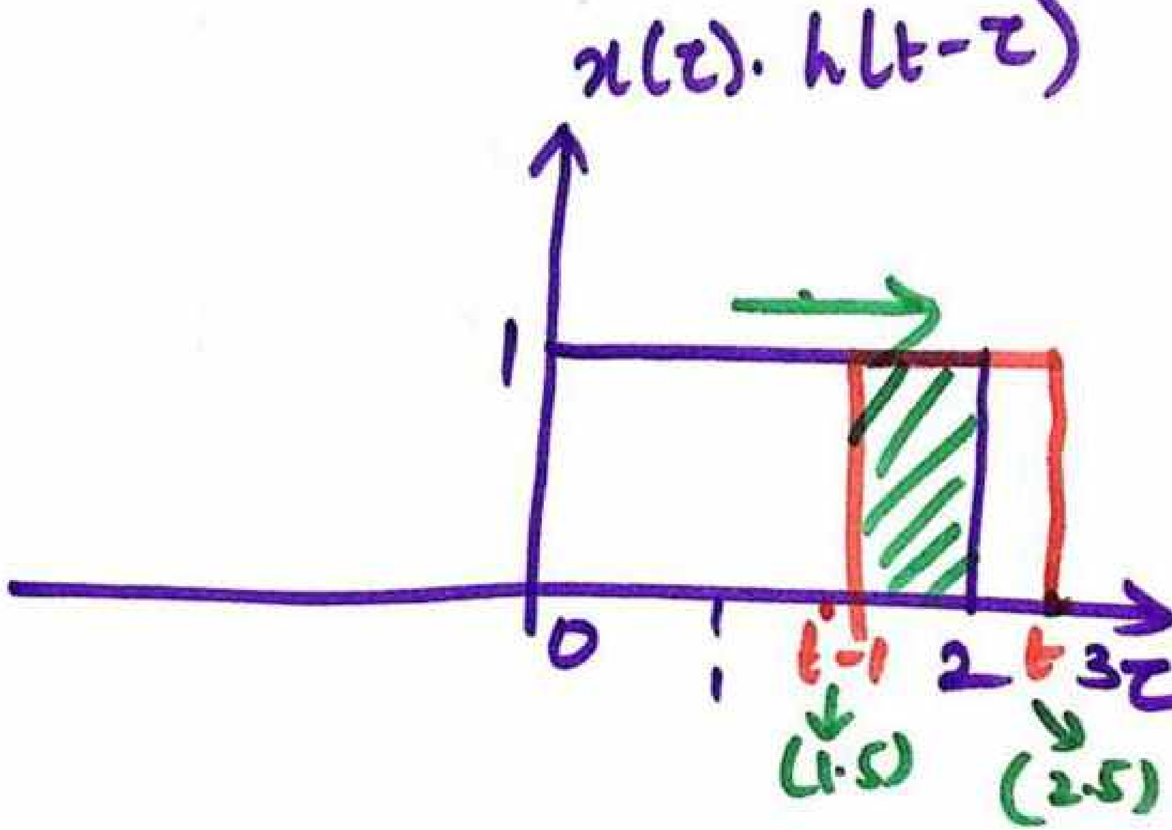


ase 4 : $2 < t < 3$

$$y(t) = \int_{t-1}^{2} 1 \, d\tau$$

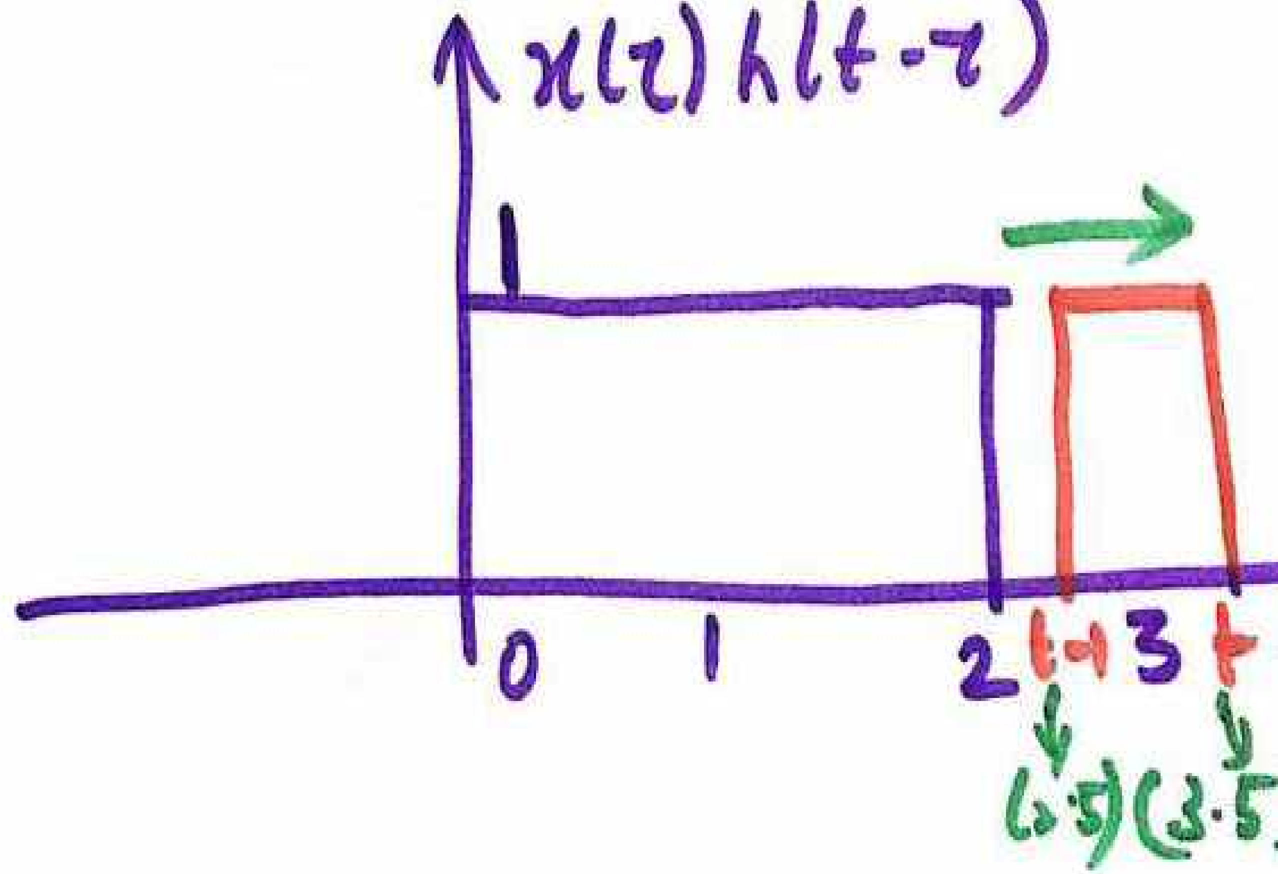
$$= \tau \Big|_{t-1}^{2}$$

$$y(t) = 2 - t + 1 = 3 - t$$

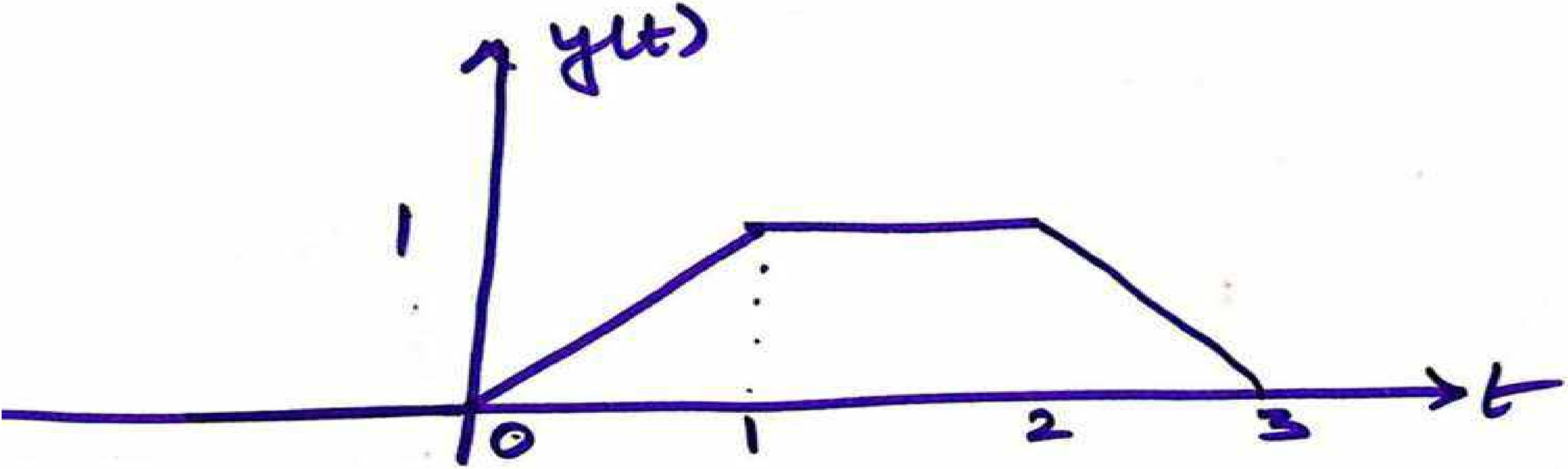


ase 5 : $t > 3$

$(t) = 0$



$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 3 - t & 2 < t < 3 \\ 0 & t > 3 \end{cases}$$



Practice question

Compute $y(t)=x(t)*h(t)$

Where $x(t)=h(t)=1$ for $0 \le t \le 1$

0 elsewhere

Properties of Convolution sum

Properties of Convolution Sum

- Distributive property
- Associative property
- Commutative property
- Time shifting property
- Convolution with an impulse

Distributive property.

$$x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

proof:

Consider two LTI systems with impulse responses $h_1[n]$ and $h_2[n]$ connected in parallel.

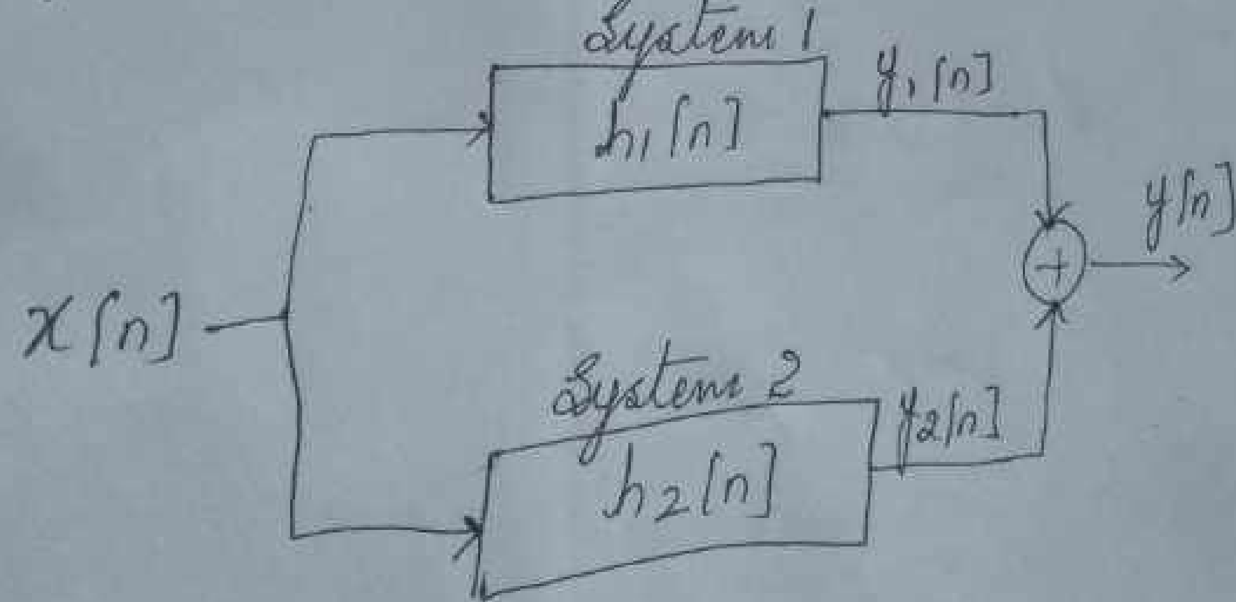
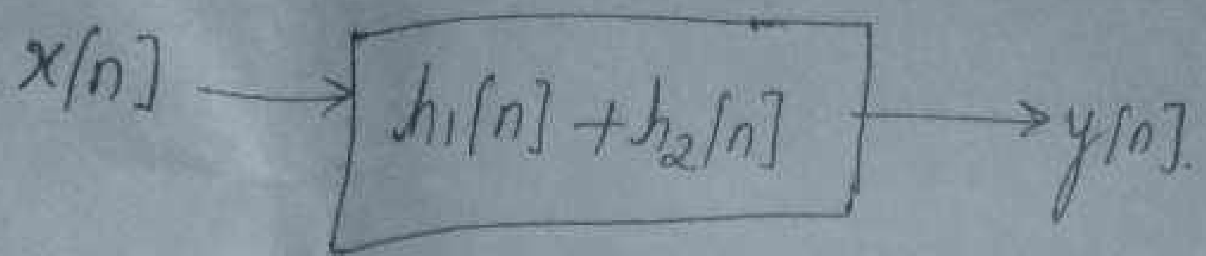


Fig 1



From Fig 1

The output of system 1 is given by

$$y_1[n] = x[n] * h_1[n]$$

Similarly,

the output of system 2 is

$$y_2[n] = x[n] * h_2[n]$$

The output $y[n] = y_1[n] + y_2[n]$

$$= \underline{x[n] * h_1[n]} + \underline{x[n] * h_2[n]}$$

$$= \sum_{k=-\infty}^{\infty} x[k] h_1[n-k] + \sum x[k] h_2[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k] \underbrace{\left[h_1[n-k] + h_2[n-k] \right]}$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$y[n] = x[n] * h[n]$$

where $h[n] = h_1[n] + h_2[n]$

Associative property

$$[x[n] * h_1[n]] * h_2[n] = x[n] * [h_1[n] * h_2[n]]$$

proof: Let us consider two LTI systems with impulse responses $h_1[n]$ and $h_2[n]$ connected in cascade.

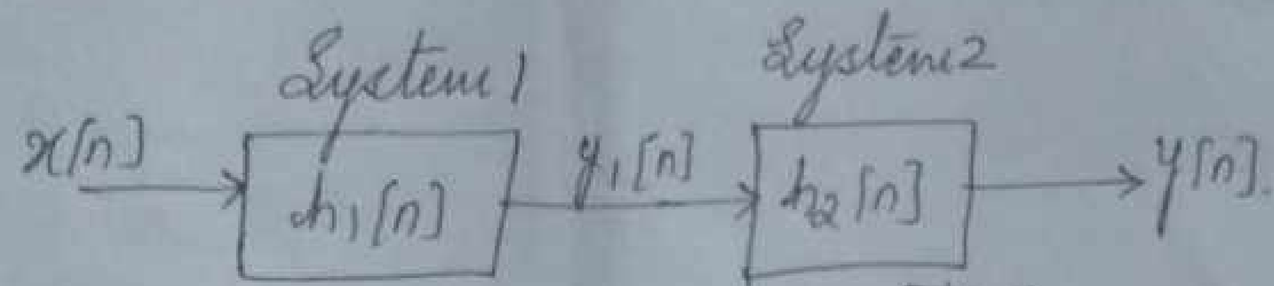


Fig. 2.

From fig. 2

Let $y_1[n]$ be the output of the first system.

Then

$$y_1[n] = x[n] * h_1[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h_1[n-k]$$

And $y[n]$ is the output of the second system.

$$y[n] = y_1[n] * h_2[n]$$

$$= \underbrace{\left[\sum_{k=-\infty}^{\infty} x[k]\, h_1[n-k]\right]}\, * h_2[n]$$

$$= \sum_{p=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[p]\, h_1[k-p]\, h_2[n-k]$$

$$y[n] = y_1[n] * h_2[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} y_1[k]\, h_2[n-k]$$

$$y_1[k] = x[k] * h_1[k]$$

$$= \sum_{p=-\infty}^{\infty} x[p]\, h_1[k-p]$$

Let

$l = k - p$, then

$$y[n] = \sum_{p=-\infty}^{\infty} x_1[p] \sum_{l=-\infty}^{\infty} h_1[l]\, h_2[n-p-l]$$

$$= \sum_{p=-\infty}^{\infty} x_1[p]\, h[n-p]$$

$$y[n] = x[n] * h[n]$$

where $h[n] = h_1[n] * h_2[n]$

Commutative property

$$x[n] * h[n] = h[n] * x[n]$$

proof: Consider

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Let $n-k = l$

$$x[n] * h[n] = \sum_{l=-\infty}^{\infty} x[n-l]\, h[l]$$

rearranging the terms

$$x[n] * h[n] = \sum_{l=-\infty}^{\infty} h[l]\, x[n-l]$$

$$x[n] * h[n] = h[n] * x[n]$$

Shifting property

If $x[n] * h[n] = y[n]$, then

$$x[n-k] * h[n-m] = y[n-k-m]$$

Proof:

Consider

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

If $x[n]$ is shifted by $l \Rightarrow x[n-l]$

$h[n]$ is —— " —— $m \Rightarrow h[n-m]$

then

$$x[n-l] * h[n-m] = \sum_{k=-\infty}^{\infty} x[k-l]\, h[(n-m)-k]$$

Let $k - l = r$

$$k = r + l$$

$$x[n-l] * h[n-m] = \sum_{r=-\infty}^{\infty} x[r]\, h[(n-m)-(r+l)]$$

$$= \sum_{r=-\infty}^{\infty} x[r]\, h[\underbrace{(n-m-l)}-r]$$

$$= y[s] \;\Big|\; s = n-m-l$$

$$x[n-l] * h[n-m] = y[n-m-l]$$

Convolution with an impulse

Convolution of a signal $x[n]$ with a unit impulse is the signal $x[n]$ itself

That is

$$x[n] * \delta[n] = x[n]$$

proof:

$$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

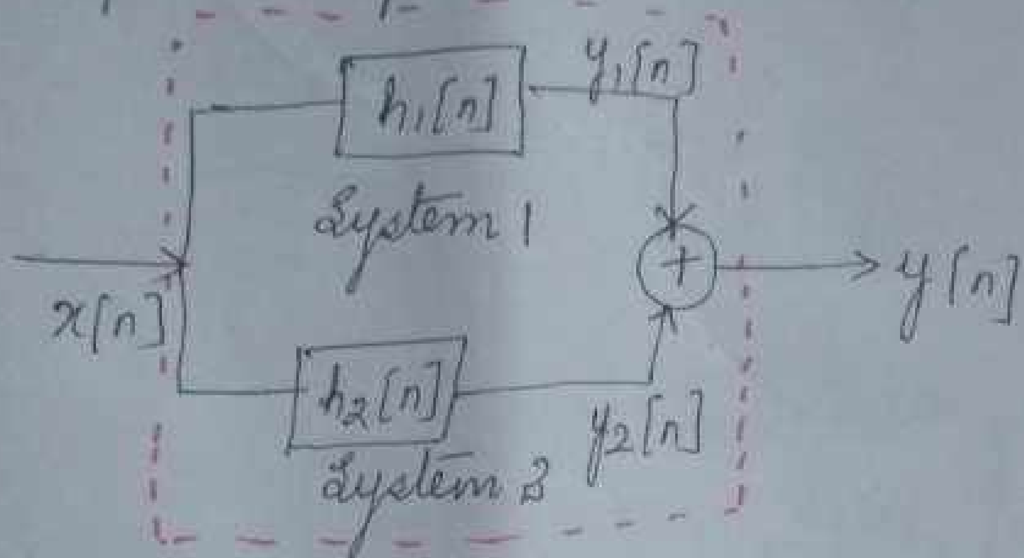
$$x[n] * \delta[n] = x[n]$$

$\delta[n-k] = 1$ for $n=k$

$\delta[n-k] = 0$ otherwise

Observations

1. Impulse response of two systems connected in parallel is sum of the individual impulse responses.



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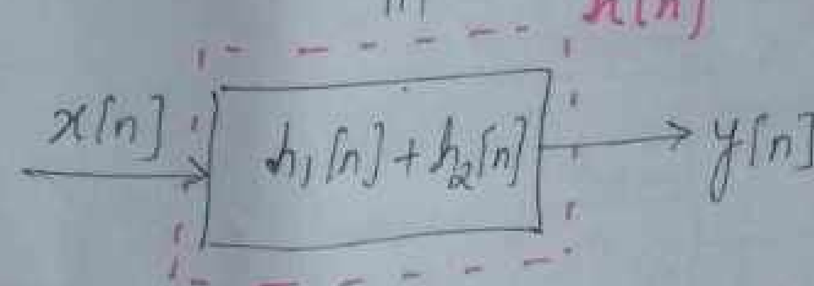
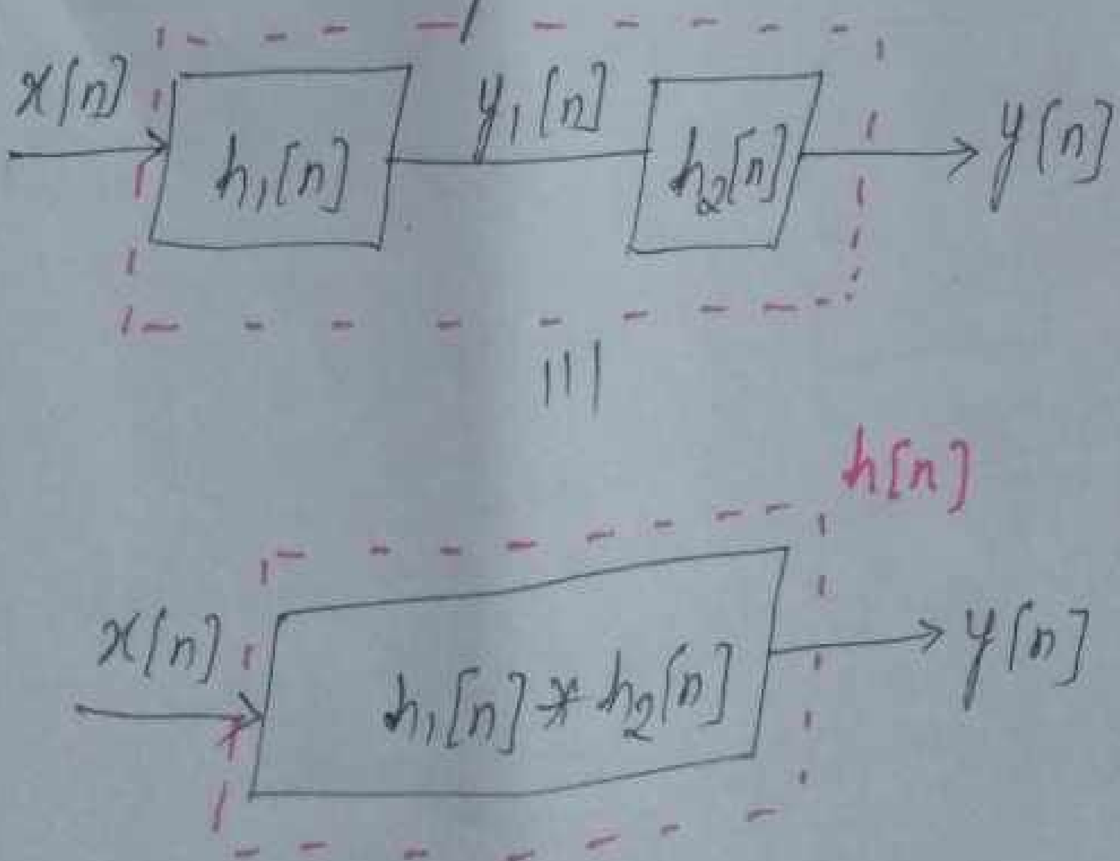
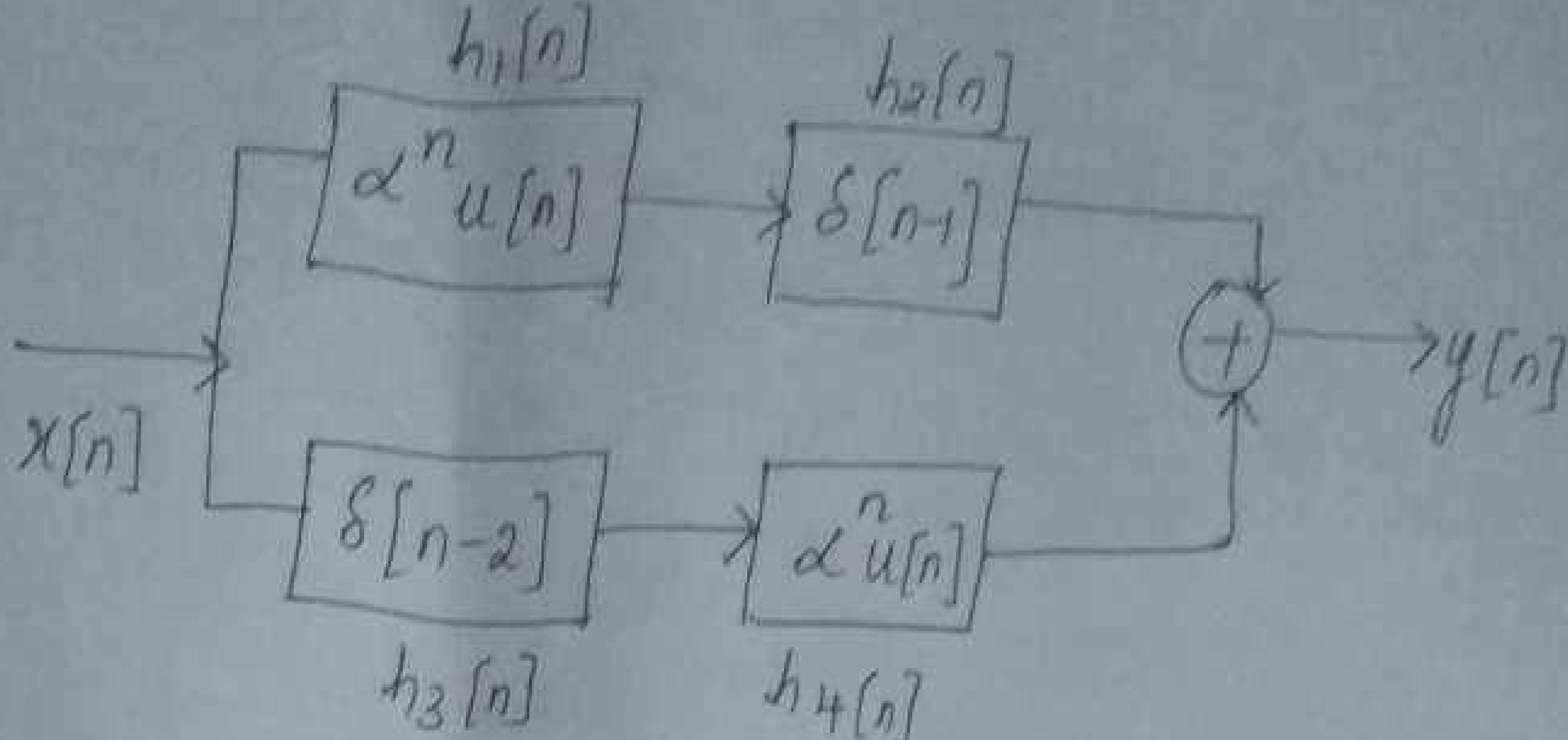


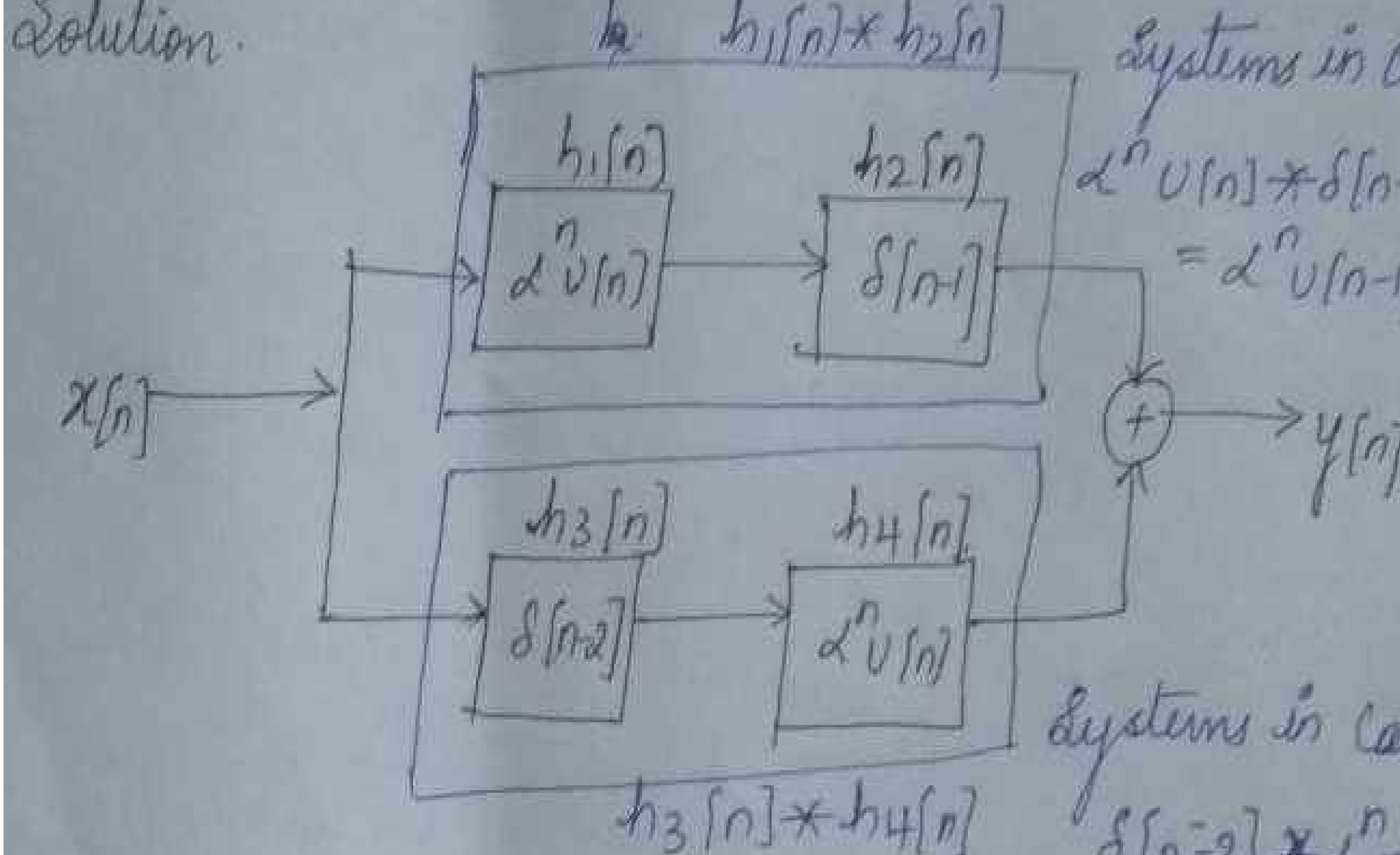
2. Impulse response of two systems connected in series is equal to the convolution of the individual responses.



Example: Find the overall impulse response of the system



Solution.

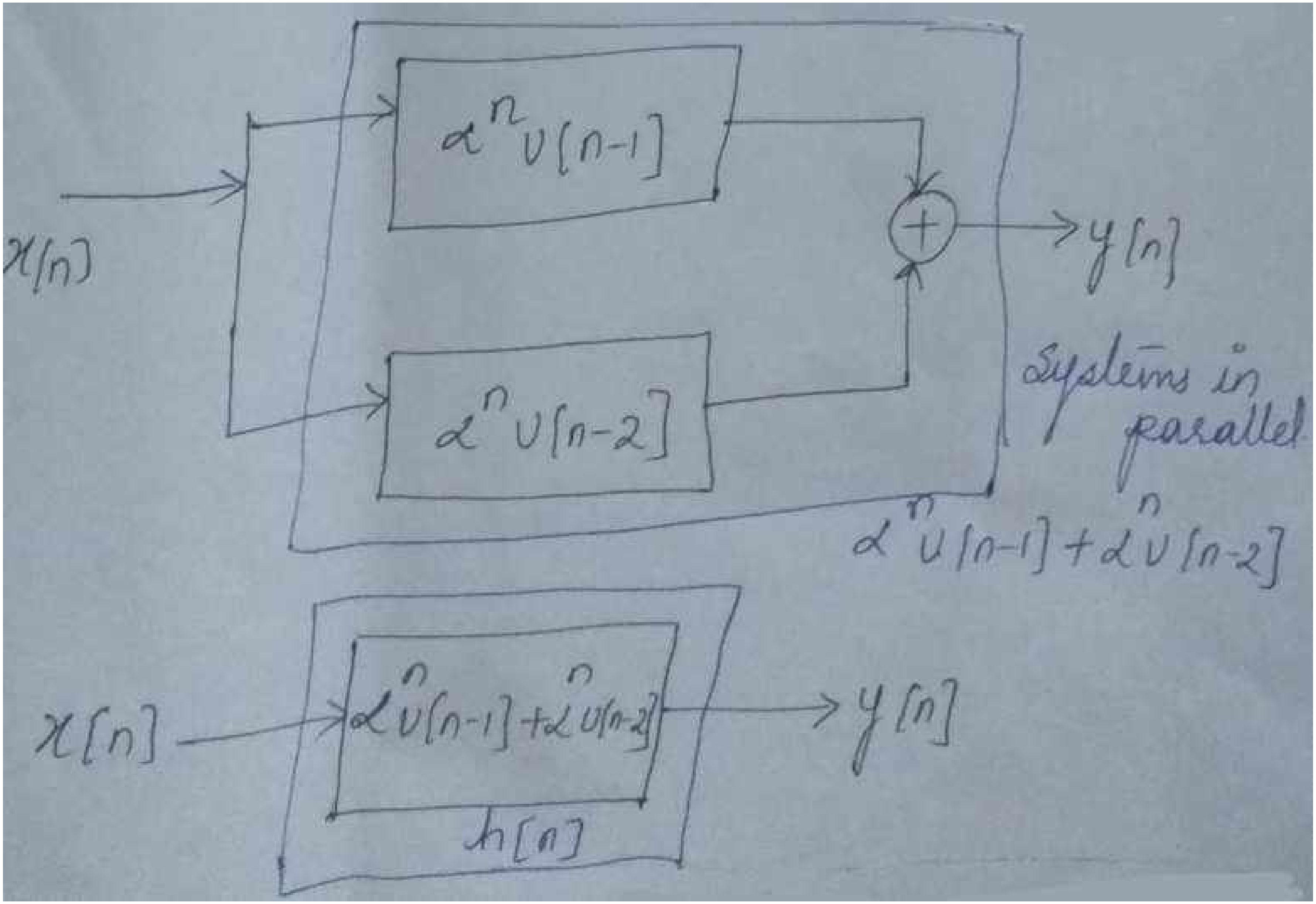


Systems in cascade.

$$\alpha^n u[n] * \delta[n-1] = \alpha^n u[n-1]$$

Systems in Cascade.

$$\delta[n-2] * \alpha^n u[n] = \alpha^n u[n-2]$$



$\alpha^n u[n-1]$
$\alpha^n u[n-2]$
$x[n]$
$y[n]$
Systems in parallel
$\alpha^n u[n-1] + \alpha^n u[n-2]$
$x[n]$
$\alpha^n u[n-1] + \alpha^n u[n-2]$
$y[n]$
$h[n]$

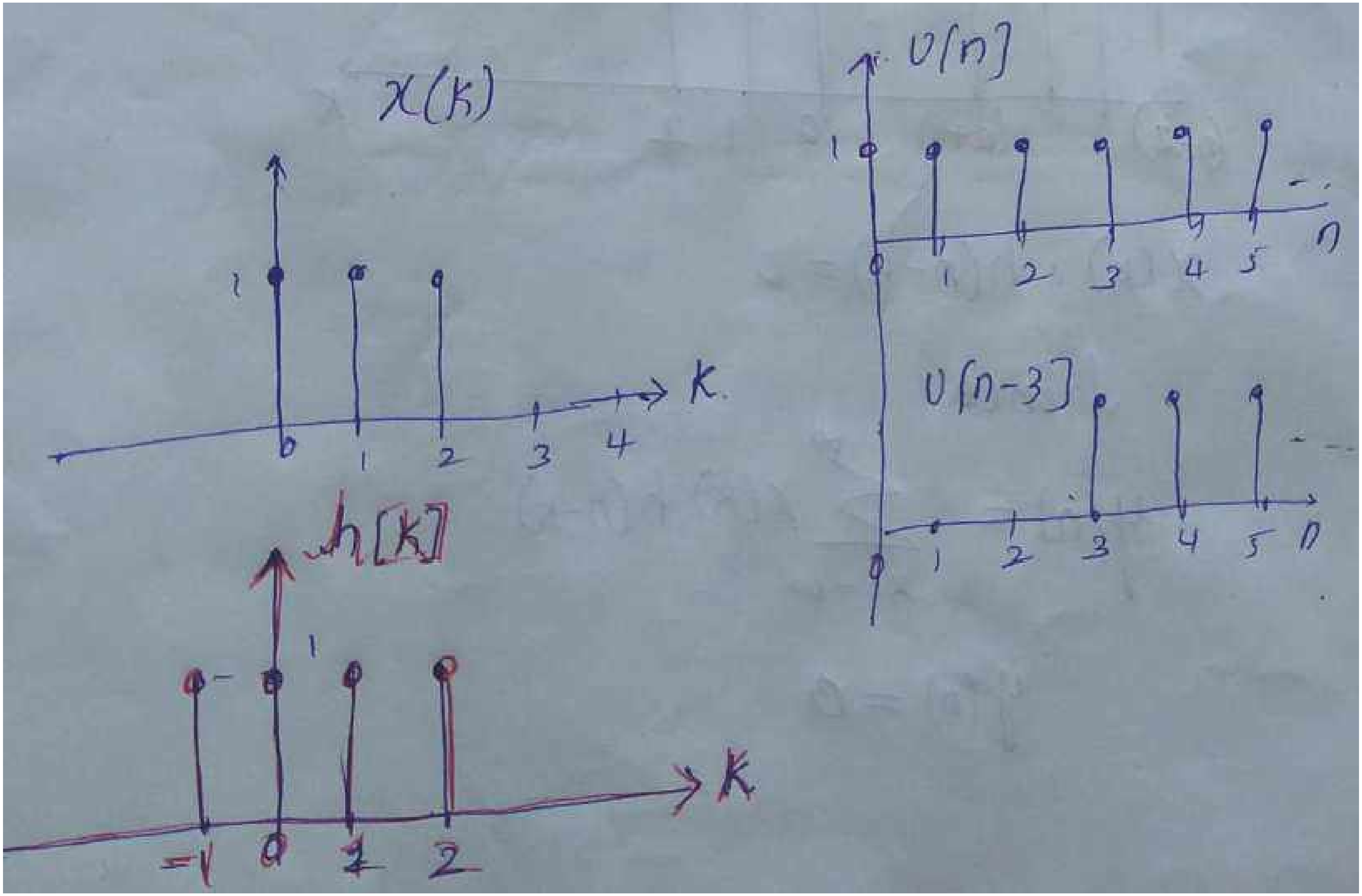
Continuation of Convolution sum(both the sequences are finite)

Example 6 : $x(n) = u(n) - u(n-3)$

$h(n) = u(n+1) - u(n-3)$

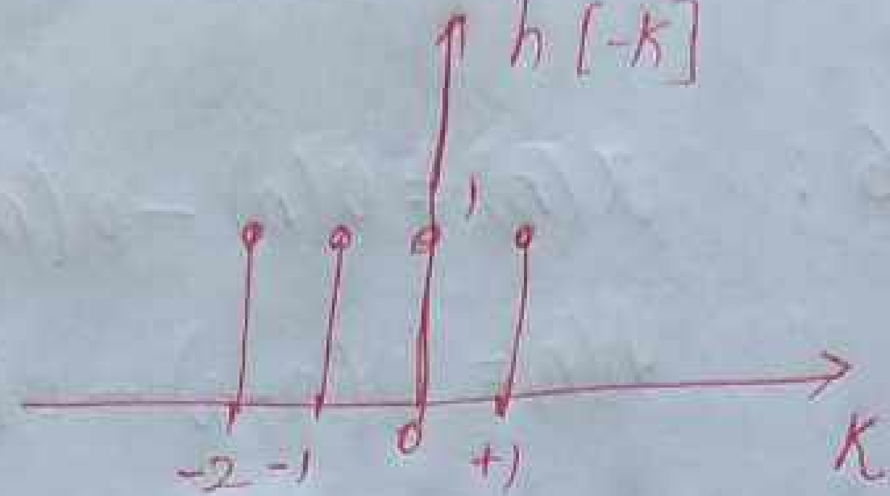
Both $x(n)$ & $h(n)$ are finite sequences

Step 1 : Sketch $x(k)$ & $h(k)$

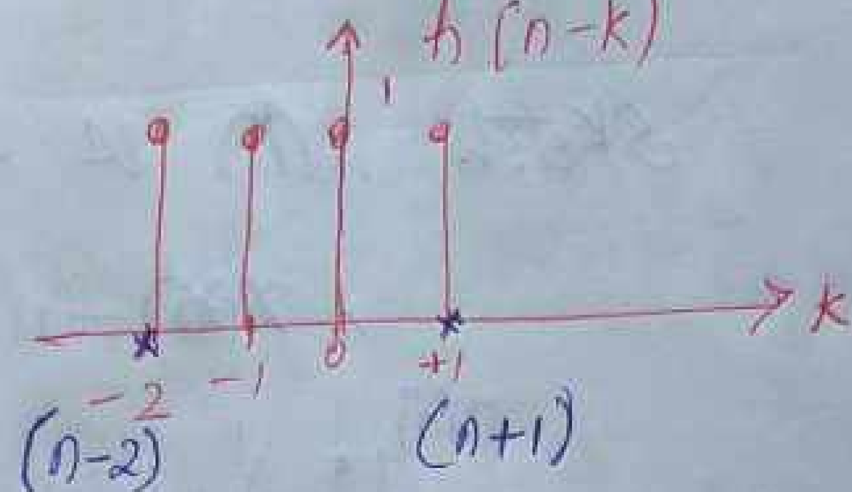


x(k)
1
0 1 2 3 4
k
h[k]
1
-1 0 1 2
k
u[n]
1
0 1 2 3 4 5
n
u[n-3]
0 1 2 3 4 5
n

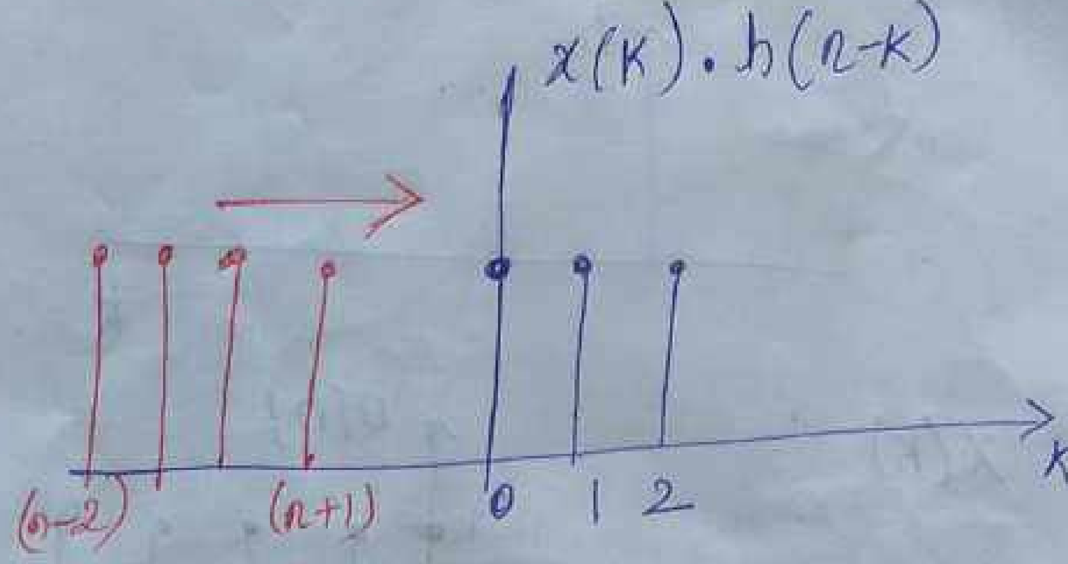
Step 2 : Sketch $h(-k)$



Step 3 : Sketch $h(n-k)$



Case 1 : $(n+1) < 0$ $[n < 1]$



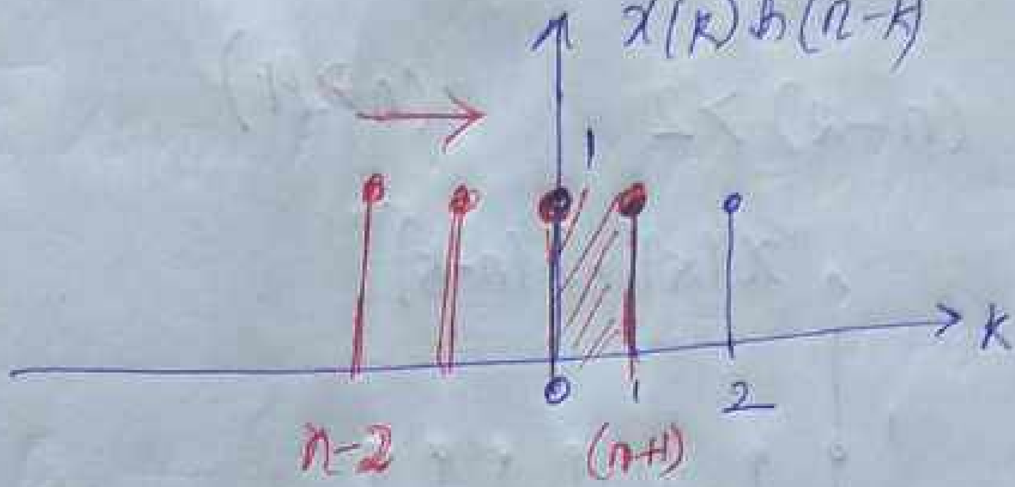
$$x(k) \cdot h(n-k) = 0$$

hence

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$y(n) = 0$$

Case 2 : $(n+1) > 0$ & $(n-2) < 0$ $[n > -1$ & $n < 2]$



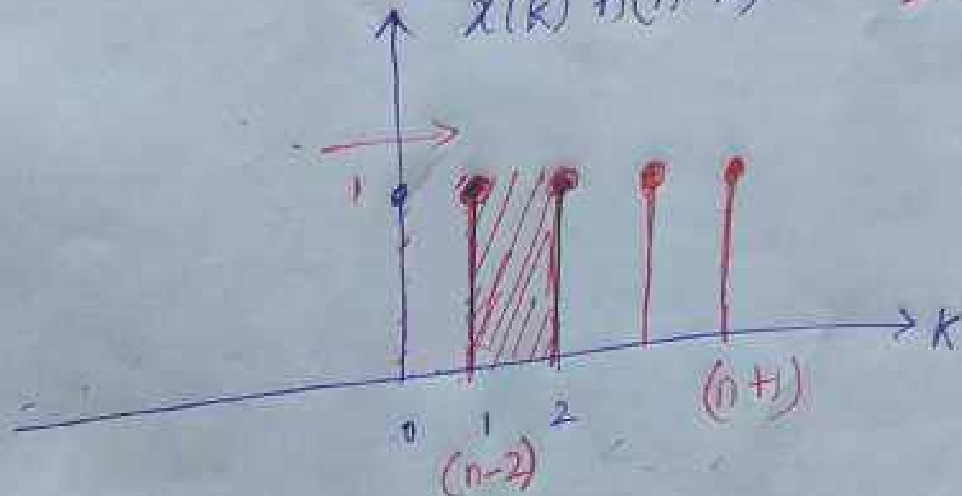
$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=0}^{n+1} 1$$

$$y[n] = n+2$$

Case 3 : $(n-2) > 0$ $(n-2) < 2$

$2 < n < 4$

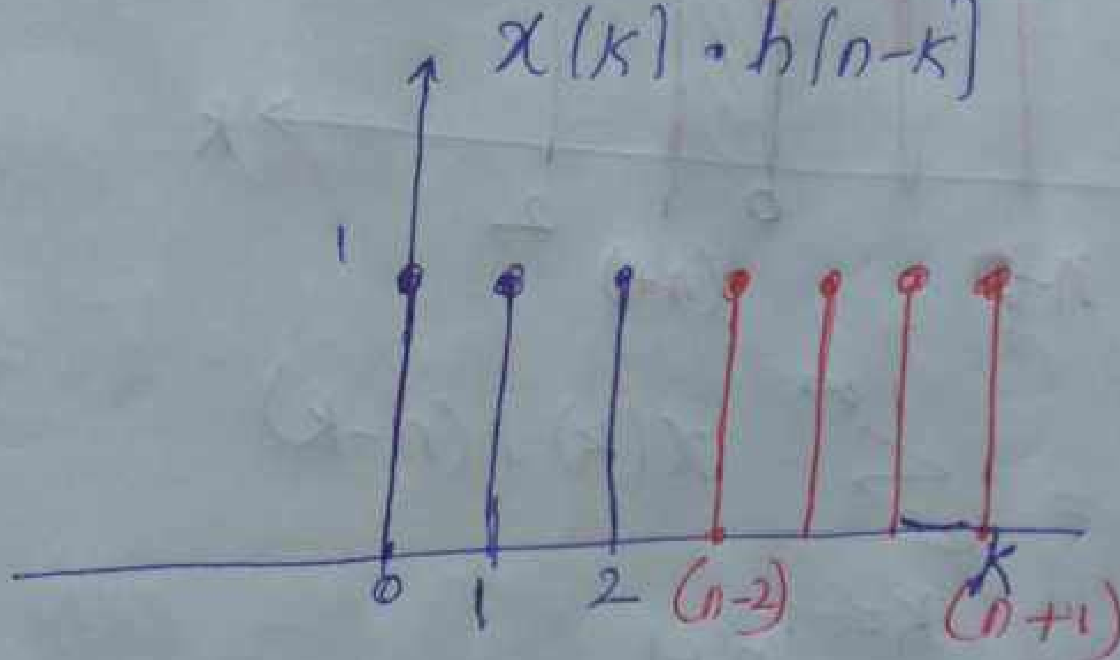


$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=(n-2)}^{2} 1 \Rightarrow$$

$$y[n] = 5-n$$

Case 4 : $(n-2) > 2$ $(n > 4)$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[n] = 0$$

ASSIGNMENT FOR TODAY

1) $x(n) = u(n) \quad h(n) = u(n-5)$

2) $x(n) = \left(\frac{1}{2}\right)^n u(n) \quad h(n) = u(n-2)$

3) $h(n) = u(n) - u(n-10)$

$x(n) = u(n-2) - u(n-7)$

Fourier Representation for Signals

→ Representing a signal as a weighted super-position of complex sinusoids.

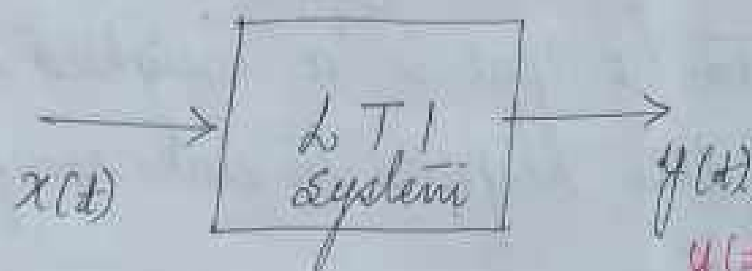
→ If such a signal is applied to a linear s/m then the system output is a weighted superposition of the system response to each complex sinusoid.

→ Provides characterization of signals and systems

→ Study of signals and systems using sinusoidal representations is termed <u>Fourier Analysis</u> after Joseph Fourier.

- Analysis of signals [spectrum]
- Helps is finding the response of LTI systems
- Digital signal processing, signal manipulation

→ When a complex sinusoid input to a LTI system, it generates an output equal to the sinusoidal input multiplied by the system frequency response.



$$y(t) = H\{x(t)\}$$

$$\boxed{x(t) = e^{j\omega t}}$$

$$\boxed{y(t) = H(j\omega)\, e^{j\omega t}}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$H(j\omega)$ — frequency response

→ Now consider expressing the input $x(t)$ to the LTI system as the weighted sum of M complex sinusoids

i.e. $$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}$$

Then each term of the input $a_k e^{j\omega_k t}$ produces an output term, $a_k H(j\omega_k) e^{j\omega_k t}$

∴ The output of the system is expressed as

$$y(t) = \sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t}$$

It is observed that the output is a weighted sum of M complex sinusoids with the weights a_k, modified by the system frequency response $H(j\omega_k)$.

Fourier Representations for four signal classes

- There are four distinct Fourier representations, each applicable to a different class of signals

Periodic Signals have Fourier series representation

Continous-time periodic signals — FS

Discrete-time periodic signals — Discrete time Fourier series (DTFS)

Nonperiodic signals have Fourier Transform representation

Continous-time & non periodic — FT

Discrete-time & non periodic — DTFT

Relationship between Time properties of a signal and the Appropriate Fourier representation

Time property	Periodic	Nonperiodic
Continous	Fourier Series (FS)	Fourier Transform (FT)
Discrete	Discrete Time Fourier Series (DTFS)	Discrete time Fourier Transform (DTFT)

Fourier Series — Continuous time periodic signals

Let $x(t)$ be periodic with period T, this can be represented by the infinite series of complex exponentials

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}}$$

Synthesis equation

where $X(k)$ — Fourier series coefficients of $x(t)$

$\omega_0 = \frac{2\pi}{T}$ rad/sec.

$$\boxed{X(k) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt}$$

Analysis equation

$\int_{\langle T\rangle}$ — integration over one period of $x(t)$.

$$\boxed{X(k) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt.}$$ Analysis equation

In $X(k)$, $X(0)$ is called the average value or dc value of the waveform.

→ A periodic waveform $x(t)$ and its Fourier coefficient $X(k)$ can be represented symbolically as

$$x(t) \xleftrightarrow{FS} X(k)$$

Existance of Fourier Series

The conditions under which a periodic signal can be represented by a Fourier series are known as Dirichlet conditions - named after the mathematician Dirichlet

They are.

1. The function $x(t)$ have only a finite number of maxima and minima.

2. The function $x(t)$ have a finite number of discontinuities.

3. The function $x(t)$ is absolutely integrable (stable).

$$\int |x(t)| dt < \infty$$

Amplitude and Phase spectra of periodic signals

The complex Fourier Coefficient is given by

$$X(k) = A(k) + jB(k)$$

The magnitude of $X(k)$ is given by

$$\boxed{|X(k)| = \sqrt{A^2(k) + B^2(k)}}$$

and the phase of $X(k)$ is given by

$$\boxed{\angle X(k) = \theta(k) = \tan^{-1}\left[\frac{B(k)}{A(k)}\right]}$$

Amplitude spectrum: The plot of $|X(k)|$ versus k

Phase spectrum: The plot of $\theta(k)$ versus k.

Since $X^*(k) = X(-k)$

$X(k) = A(k) + jB(k)$

$X^*(k) = A(k) - jB(k)$

we have

$$|X(-k)| = |X(k)|$$

Amplitude spectrum is an even function of k

$$\theta(-k) = -\theta(k)$$

Phase spectrum is an odd function of k for a periodic signal.

PROPERTIES OF FOURIER SERIES

- There are 8 Properties

1. Linearity
2. Time shifting
3. Frequency shifting
4. Time Differentiation
5. Parseval’s theorem
6. Convolution
7. Modulation
8. Time Scaling

1) Linearity:

If $x(t) \xleftrightarrow{FS} X(k)$

and $y(t) \xleftrightarrow{FS} Y(k)$

then $a\,x(t) + b\,y(t) \xleftrightarrow{FS} a\,X(k) + b\,Y(k)$

Proof: We have

$$X(k) = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$Y(k) = \frac{1}{T} \int_{\langle T \rangle} y(t)\, e^{-jk\omega_0 t}\, dt$$

$$\therefore \quad Z(k) = \frac{1}{T}\int_{\langle T\rangle} z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} \left[a x(t) + b y(t)\right] e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\, a \int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\, b \int_{\langle T\rangle} y(t)\, e^{-jk\omega_0 t}\, dt$$

$$Z(k) = a X(k) + b Y(k)$$

Hence prooved

2) Time Shift:

If $x(t) \xleftrightarrow{FS} X(K)$

then $\underbrace{x(t-t_0)}_{y(t)} \xleftrightarrow{FS} \underbrace{e^{-jK\omega_0 t_0} X(K)}_{Y(K)}$

Proof: w.r.t.

$$X(K) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jK\omega_0 t}\, dt$$

$$\therefore Y(K) = \frac{1}{T}\int_{\langle T\rangle} y(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t-t_0)\, e^{-jK\omega_0 t}\, dt$$

Let $(t-t_0)=m \Rightarrow t=t_0+m$

$$Y(k)=\frac{1}{T}\int_{\langle T\rangle} x(m)\,e^{-jk\omega_0(t_0+m)}\,dm$$

$$Y(k)=\frac{1}{T}\int_{\langle T\rangle} x(m)\,e^{-jk\omega_0 m}\cdot e^{-jk\omega_0 t_0}\,dm$$

$$Y(k)=e^{-jk\omega_0 t_0}\,\frac{1}{T}\int_{\langle T\rangle} x(m)\,e^{-jk\omega_0 t_0}\,dm$$

$$\boxed{Y(k)=e^{-jk\omega_0 t_0}\,X(k)}$$

3) Frequency shift:

If $x(t) \xleftrightarrow{FS} X(k)$

then $\underbrace{e^{jk_0\omega_0 t}\, x(t)}_{y(t)} \xleftrightarrow{FS} \underbrace{X(k-k_0)}_{Y(k)}$

Proof: we have $X(k) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$

$$Y(k) = \frac{1}{T}\int_{\langle T\rangle} y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{jk_0\omega_0 t}\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-j\omega_0 t(k-k_0)}\, dt$$

$$\boxed{Y(k) = X(k-k_0)}$$

Time Differentiation

4) If $x(t) \xleftrightarrow{FS} X(k)$

then $\frac{d}{dt}x(t) \xleftrightarrow{FS} jk\omega_0 X(k)$

Proof: We have $x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$

Differentiate both the sides w.r.t time t

we get

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}\right]$$

changing the order of differentiation and summation

$$\frac{d\,x(t)}{dt} = \sum_{k=-\infty}^{\infty} X(k) \frac{d}{dt} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \cdot jk\omega_0$$

$$\frac{d\,x(t)}{dt} = \sum_{k=-\infty}^{\infty} \left[X(k)\, jk\omega_0\right] e^{jk\omega_0 t}$$

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{FS} jk\omega_0 X(k)}$$

5) Convolution:

If $x(t) \xleftrightarrow{FS} X(k)$

$y(t) \xleftrightarrow{FS} Y(k)$

then $\underline{x(t) \circledast y(t)} \xleftrightarrow{FS} T \cdot X(k) Y(k)$

Proof

periodic / circular convolution

$z(t)$

$$Z(k) = \frac{1}{T} \int_{\langle T \rangle} z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{\langle T \rangle} x(t) \circledast y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle}\int_{l=\langle T\rangle} x(l)\, y(t-l)\, dl\, e^{-jk\omega_0 t}\, dt$$

Rearranging the terms

$$Z(k) = \frac{1}{T}\int_{l=\langle T\rangle} x(l) \int_{t=\langle T\rangle} y(t-l)\, e^{-jk\omega_0 t}\, dt\, dl$$

Let $t-l=m \Rightarrow dt = dm$

$$Z(k) = \frac{1}{T}\int_{l=\langle T\rangle} x(l) \int_{m=\langle T\rangle} y(m)\, e^{-jk\omega_0 (m+l)}\, dm\, dl$$

$$= \underbrace{\frac{1}{T}\int_{l=\langle T\rangle} x(l)\, e^{-jk\omega_0 l}\, dl}_{TX(k)}\ \underbrace{\int_{m=\langle T\rangle} y(m)\, e^{-jk\omega_0 m}\, dm}_{TY(k)}$$

$$\boxed{Z(k) = T\, X(k)\, Y(k)}$$

6) Modulation:

If $x(t) \xleftrightarrow{FS} X(k)$

$y(t) \xleftrightarrow{FS} Y(k)$

then $\underbrace{x(t)\cdot y(t)}_{z(t)} \xleftrightarrow{FS} \underbrace{X(k) * Y(k)}_{Z(k)}$

Proof: We have $Z(k) = \frac{1}{T}\int_{\langle T\rangle} z(t)\, e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{T}\int_{\langle T\rangle} (x(t)\cdot y(t))\, e^{-jk\omega_0 t}\, dt$$

Substituting for $x(t)$ $\left\langle x(t) = \sum_{l=-\infty}^{\infty} X(l)\, e^{jl\omega_0 t}\right.$

$$Z(k) = \frac{1}{T}\int_{\langle T\rangle}\left[\sum_{l=-\infty}^{\infty} X(l)\, e^{jl\omega_0 t}\right] y(t)\, e^{-jk\omega_0 t}\, dt$$

Changing the order of summation and integration, we get

$$Z(k) = \frac{1}{T}\left[\sum_{l=-\infty}^{\infty} X(l) \int_{\langle T\rangle} y(t)\, e^{-j(k-l)\omega_0 t}\, dt\right]$$

$$= \sum_{l=-\infty}^{\infty} X(l)\, Y(k-l)$$

$$\boxed{Z(k) = X(k) * Y(k)}$$

7) Parseval's Theorem

If $x(t) \xleftrightarrow{FS} X(k)$

then $$\frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

Proof: The equation is the average power of a periodic continuous-time signal $x(t)$ with fundamental period T

ie $$P = \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 \, dt$$

it can be written as

$$P = \frac{1}{T}\int_{\langle T\rangle} x(t)\, x^*(t)\, dt = \frac{1}{T}\int_{\langle T\rangle} x(t) \left[\sum_{k=-\infty}^{\infty} X^*(k) e^{-jk\omega_0 t}\right] dt$$

changing the order summation and integration, we get

$$P = \frac{1}{T} \sum_{k=-\infty}^{\infty} X^*(k) \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} X^*(k) \cdot X(k)$$

$$= \sum_{k=-\infty}^{\infty} |X(k)|^2$$

$$\boxed{\therefore \frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X(k)|^2}$$

In the above eqn, $|X(k)|^2$ for $k = 0, 1, 2 \cdots$ signifies the distribution of power as a function of frequency and is called power density spectrum of the signal

8> Time Scaling

if $x(t) \xleftrightarrow{FS} X(k)$

then $z(t) = x(at) \xleftrightarrow{FS} X(k) = Z(k)$; $a > 0$

Proof: If $x(t)$ is periodic, $x(at)$ is also periodic

If $x(t)$ has fundamental period 'T', then

$z(t) = x(at)$ has fundamental period $\left(\frac{T}{a}\right)$.

On the other hand, if $x(t)$ has a fundamental frequency equal to ω_0, then the fundamental frequency of $z(t)$ is $a\omega_0$.

We have,
$$X(k) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$Z(k) = \frac{1}{T/a}\int_{\langle T/a\rangle} z(t)\, e^{-jka\omega_0 t}\, dt$$

$$= \frac{a}{T}\int_{\langle T/a\rangle} x(at)\, e^{-jka\omega_0 t}\, dt$$

Let $at = l$

then $dt = \frac{1}{a}\, dl$

$$Z(k) = \frac{a}{T}\int_{\langle T/a\rangle} x(l)\, e^{-jk\omega_0 l}\, \frac{dl}{a}$$

$$Z(k) = \frac{1}{T}\int_{\langle T\rangle} x(l)\, e^{-jk\omega_0 l}\, dl \Rightarrow \boxed{Z(k) = X(k)}$$

Symmetry :

If $x(t) \xleftrightarrow{FS} X(k)$

then (i) $x(t)$: real $\xleftrightarrow{FS}$ $X^*(k) = X(-k)$

(ii) $x(t)$: real & even $\xleftrightarrow{FS}$ $\text{Img}\{X(k)\} = 0$

(iii) $x(t)$: real and odd $\xleftrightarrow{FS}$ $\text{Re}\{X(k)\} = 0$

Example 1: For the signal $x(t) = \sin\omega_0 t$, find the Fourier series and draw its spectrum

Solution: Given $x(t) = \sin\omega_0 t$.

$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\therefore\ x(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

Comparing this with the equation.

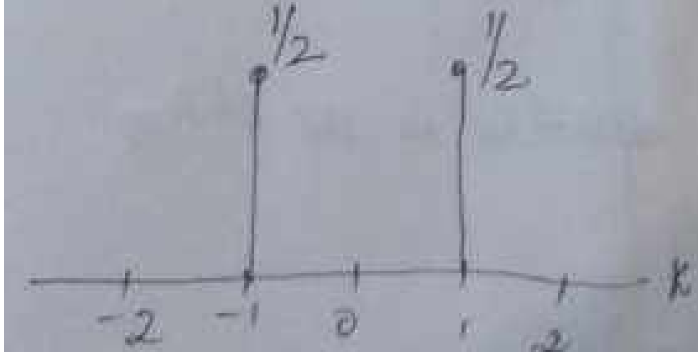
$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$$

$$x(t) = X(-1)\, e^{-j\omega_0 t} + X(1)\, e^{j\omega_0 t}$$

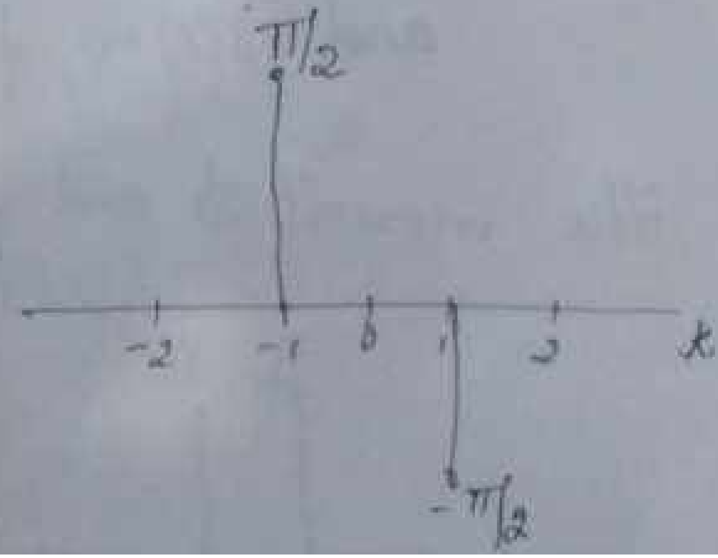
we get $X(1) = \frac{1}{2j}$ $\quad X(-1) = -\frac{1}{2j}$

$X(k) = 0$ for $k \neq \pm 1$

$|X(k)|$



$\angle X(k)$



Example 2: Evaluate the FS representation for the signal $x(t) = \sin(2\pi t) + \cos(3\pi t)$. Sketch the magnitude and phase spectra

Solution: Given $x(t) = \underline{\sin(2\pi t)} + \underline{\cos(3\pi t)}$

$\omega_{01} = 2\pi \qquad \omega_{02} = 3\pi$

$\frac{2\cancel{\pi}}{T_1} = 2\cancel{\pi} \qquad \frac{2\cancel{\pi}}{T_2} = 3\cancel{\pi}$

$T_1 = 1 \qquad T_2 = \frac{2}{3}$

$\frac{T_1}{T_2} = \text{rational}$

$x(t)$ can be written as

$$x(t) = \frac{1}{2j} e^{j2\pi t} - \frac{1}{2j} e^{-j2\pi t} + \frac{1}{2} e^{j3\pi t} + \frac{1}{2} e^{-j3\pi t}$$

Comparing this with

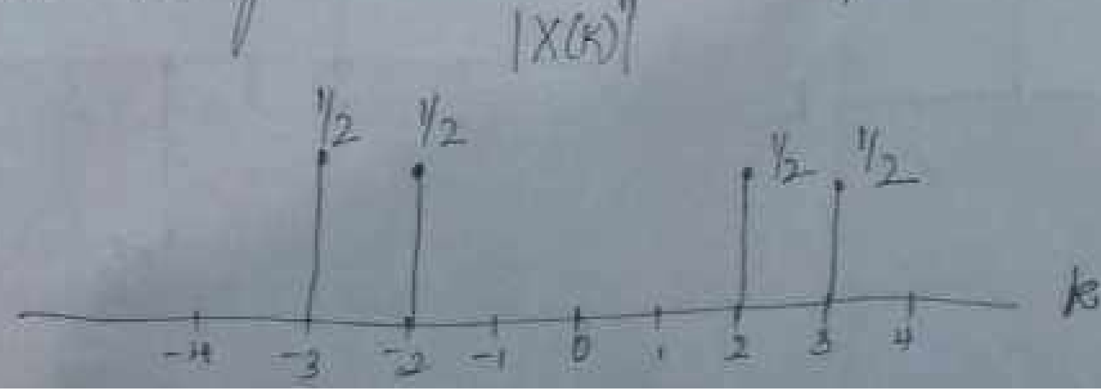
$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$$

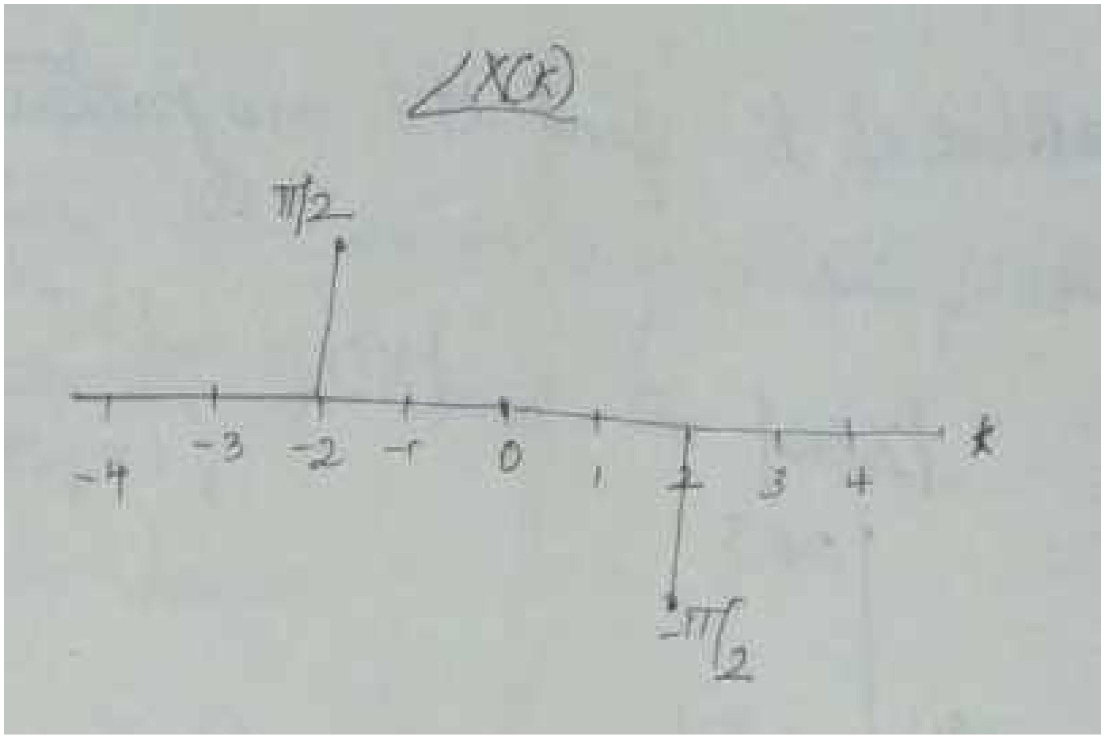
$$X(2) = \frac{1}{2j} \qquad X(-2) = \frac{-1}{2j}$$

$$X(3) = \frac{1}{2} \qquad X(-3) = \frac{1}{2}$$

and $X(k) = 0$ for $k \neq \pm 2, \pm 3$

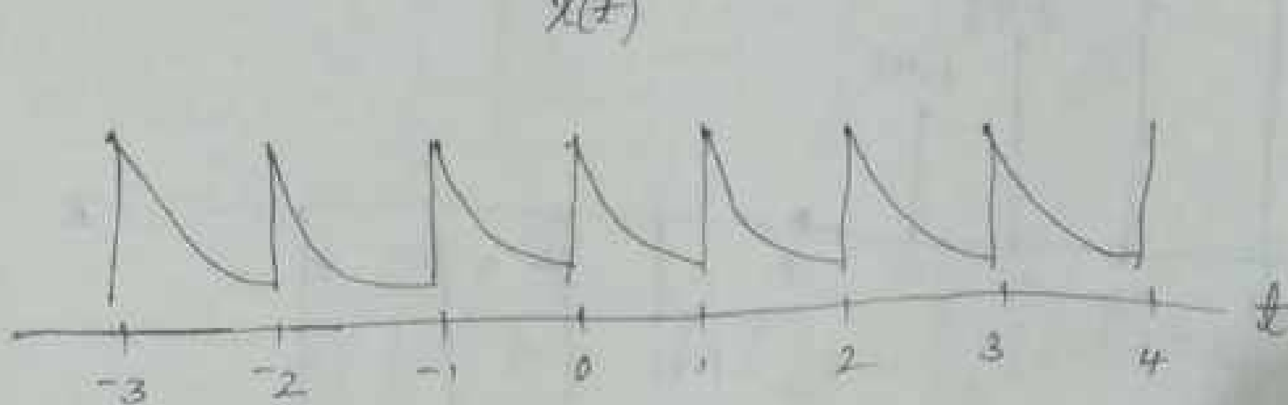
The magnitude and phase spectra is as shown





∠X(k)
π/2
-4
-3
-2
-1
0
1
2
3
4
k
-π/2

Example 3: For the signal $x(t)$ shown, find the FS representation and draw its magnitude and phase spectra



Solution: $T=1$; $\omega_0 = \frac{2\pi}{T} = 2\pi$

we have

$$X(k) = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\therefore \quad X(k) = \frac{1}{1} \int_{t=0}^{1} e^{-t}\, e^{-jk(2\pi)t}\, dt$$

$$= \int_{0}^{1} e^{-(1+j2\pi k)t}\, dt$$

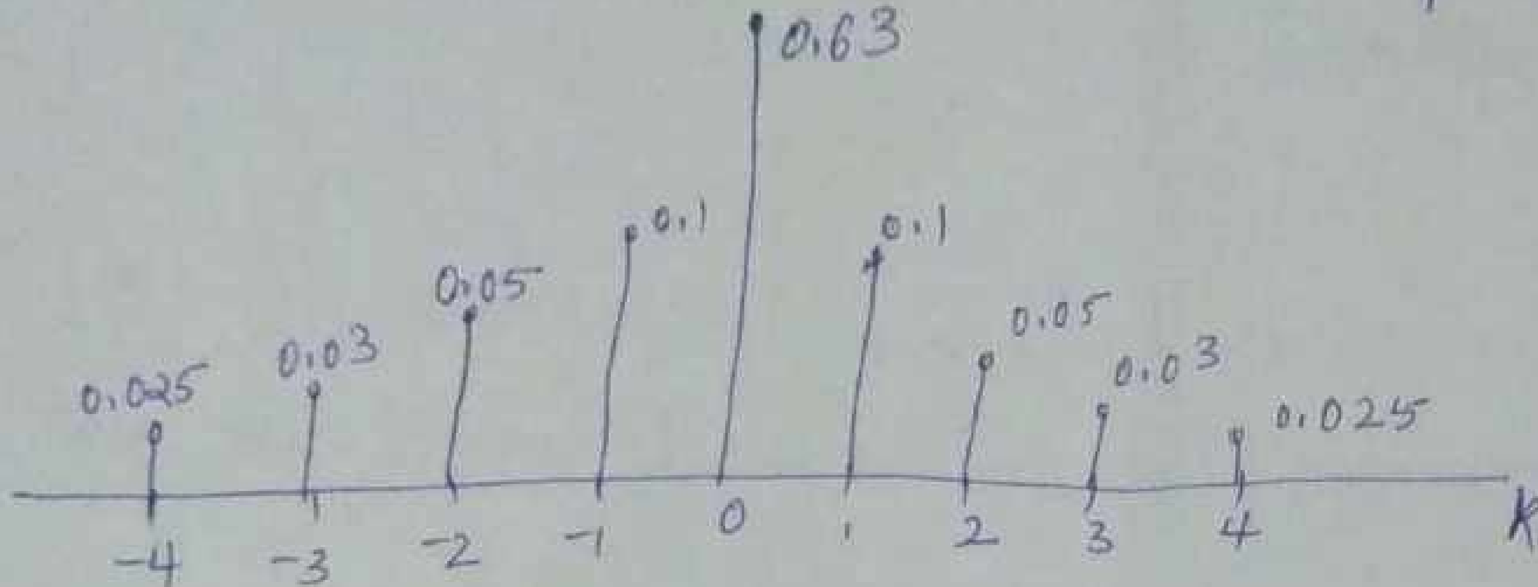
$$= \left. \frac{-e^{-(1+j2\pi k)t}}{(1+j2\pi k)} \right]_0^1$$

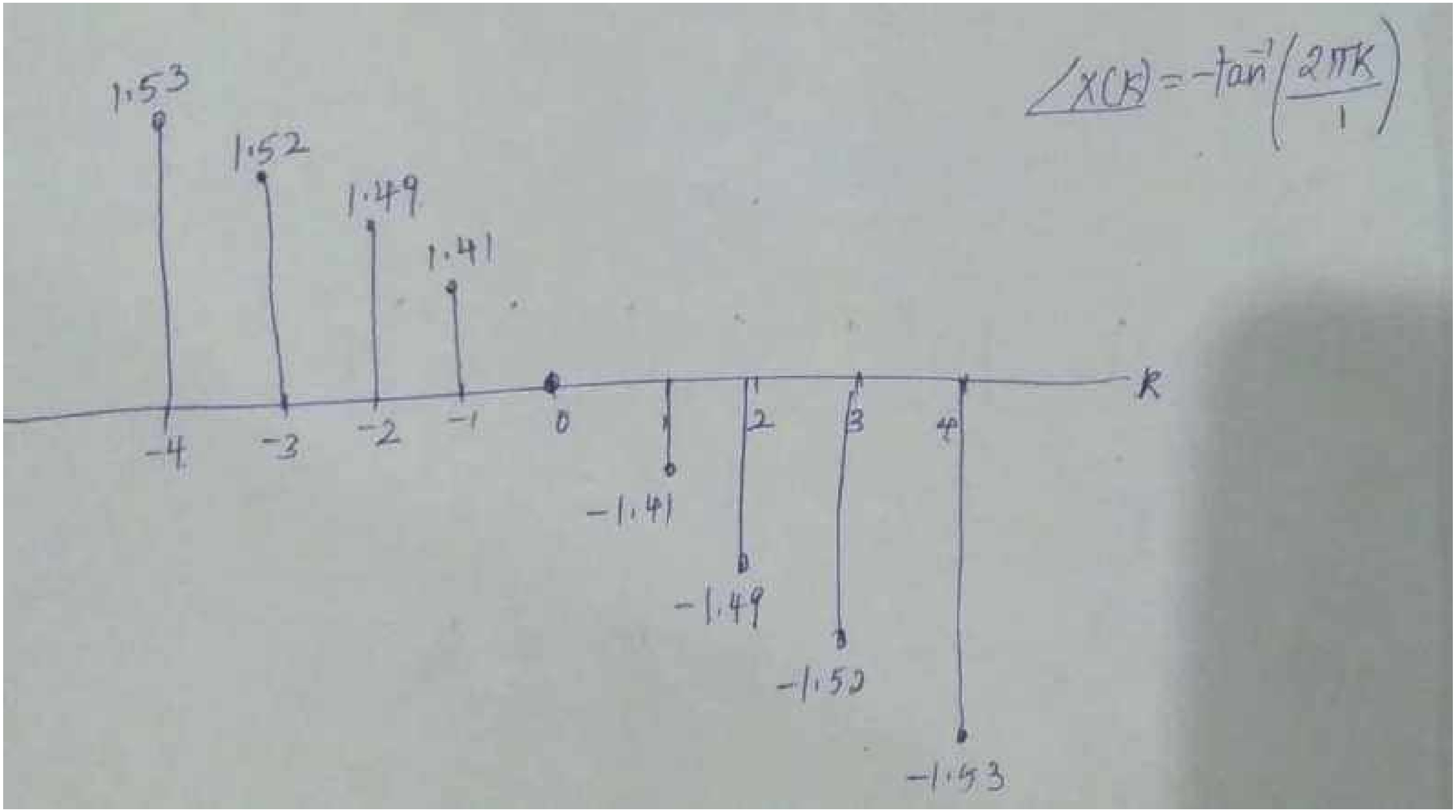
$$X(k) = \frac{1-e^{-1}}{1+j2\pi k}.$$

For each value of k, find the magnitude $|X(k)|$ & phase $\angle X(k)$.

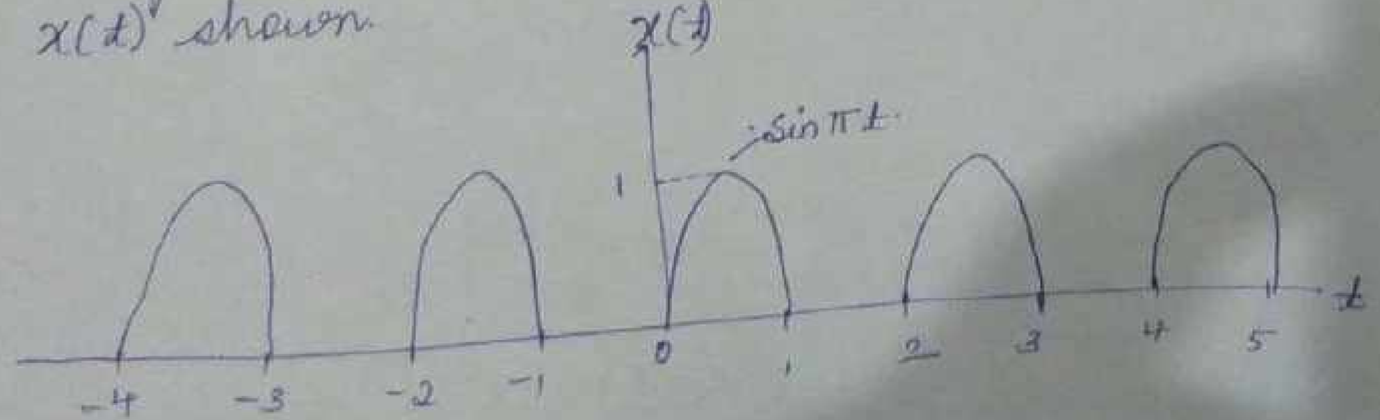
$$|X(k)| = \frac{1 - e^{-1}}{\sqrt{1 + (2\pi k)^2}}$$

$|X(k)|$




$\angle X(k) = -\tan^{-1}\left(\frac{2\pi k}{1}\right)$
1.53
1.52
1.49
1.41
-4
-3
-2
-1
0
1
2
3
4
k
-1.41
-1.49
-1.52
-1.53

Example 4: Find the FS coefficients for the signal $x(t)$ shown.



Solution: The given signal is periodic,

$T = 2$ and $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

We have.

$$X(k) = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$X(k) = \frac{1}{2} \int_0^1 \sin \pi t \; e^{-jk\omega_0 t}\, dt.$$

$$= \frac{1}{4j}\left[\int_0^1 e^{j(1-k)\pi t}\,dt - \int_0^1 e^{-j(1+k)\pi t}\,dt\right]$$

$$= \frac{1}{4j}\left[\left.\frac{e^{j(1-k)\pi t}}{j(1-k)\pi}\right]_0^1 + \left.\frac{e^{-j(1+k)\pi t}}{j(1+k)\pi}\right]_0^1\right.$$

$$= \frac{1}{4j}\left[\frac{e^{j(1-k)\pi} - e^{0}}{j(1-k)\pi} + \frac{e^{-j(1+k)\pi} - e^{0}}{j(1+k)\pi}\right]$$

$$= \frac{1}{4j}\left[\frac{(-1)^{k+1} - 1}{j(1-k)\pi} + \frac{(-1)^{k+1} - 1}{j(1+k)\pi}\right]$$

$$X(k) = \frac{-1}{4\pi}\left[(-1)^{k+1} - 1\right]\left[\frac{2}{1-k^2}\right] = \frac{1}{4\pi}\left[1-(-k)^{k+1}\right]\left[\frac{2}{1-k^2}\right]$$

Example 5. Find the FS coefficients for the periodic signal $x(t)$ with period 2 given by

$$x(t) = e^{-t} \quad ; \text{ for } -1 < t < 1$$

Solution: Given $T = 2$ $\quad \therefore \omega_0 = \frac{2\pi}{T} = \pi$

$$X(k) = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

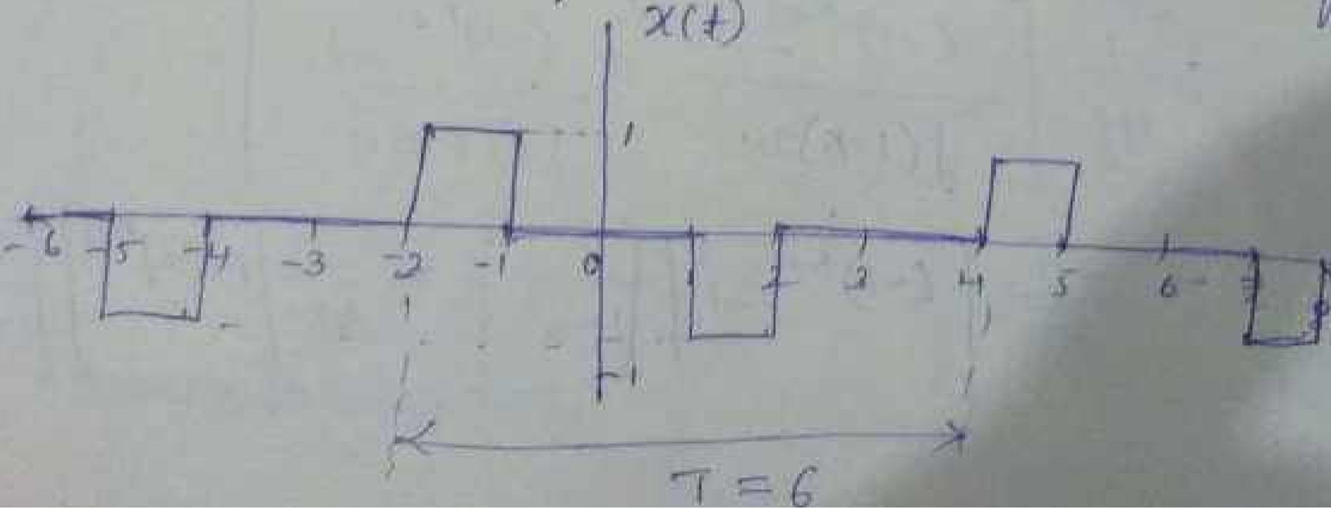
$$\therefore X(k) = \frac{1}{2} \int_{-1}^{1} e^{-t}\, e^{-jk\pi t}\, dt$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1+jk\pi)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(1+jk\pi)t}}{-(1+jk\pi)}\right]_{-1}^{1} = \frac{-1}{2(1+jk\pi)}\left[e^{-(1+jk\pi)} - e^{(1+jk\pi)}\right]$$

$$X(k) = \frac{(-1)^k}{2(1+jk\pi)}\left(e - e^{-1}\right)$$

Example 6 : Find the Fourier series coefficients of the periodic signal $x(t)$ shown in the fig



Solution: $T = 6 \quad \therefore \omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$

W.k.t $\quad X(k) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$

$$X(k) = \frac{1}{6}\int_{-3}^{3} x(t)\, e^{-jk(\pi/3)t}\, dt$$

$$= \frac{1}{6}\left[\int_{-2}^{-1} e^{-jk(\pi/2)t}\, dt - \int_{1}^{2} 1 \cdot e^{-jk(\pi/3)t}\, dt\right]$$

$$= \frac{1}{6}\left\{\left[\frac{e^{-jk(\pi/3)t}}{-jk(\pi/3)}\right]_{-2}^{1} - \left[\frac{e^{-jk(\pi/3)t}}{-jk(\pi/3)}\right]_{1}^{2}\right\}$$

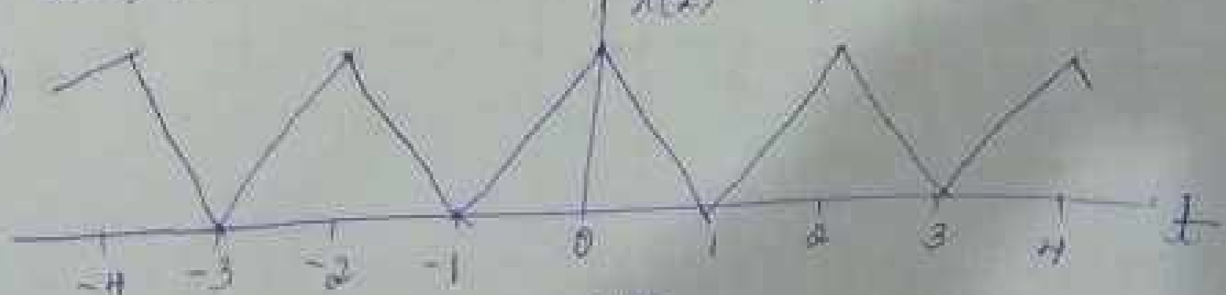
$$X(k) = \frac{1}{3}\left\{\frac{\cos\left(\frac{2}{3}\pi k\right) - \cos\left(\frac{\pi}{3}k\right)}{\left(jk\pi/3\right)}\right\}$$

Assignment

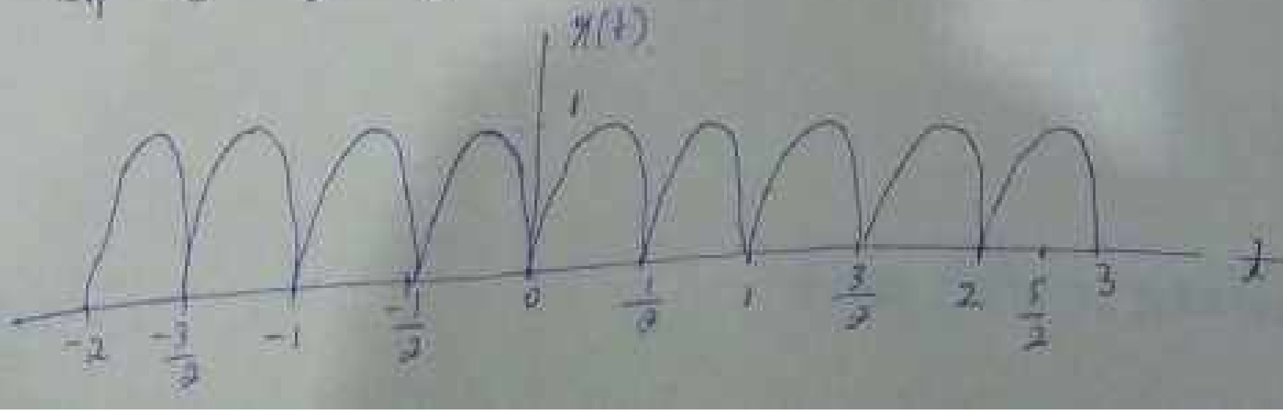
1. Determine the FS representation for the signal $x(t) = \cos 4t + \sin 8t$. Sketch the magnitude & phase spectra.

2. Find the Fourier Series representation of the signal $x(t)$ shown and draw the spectra.

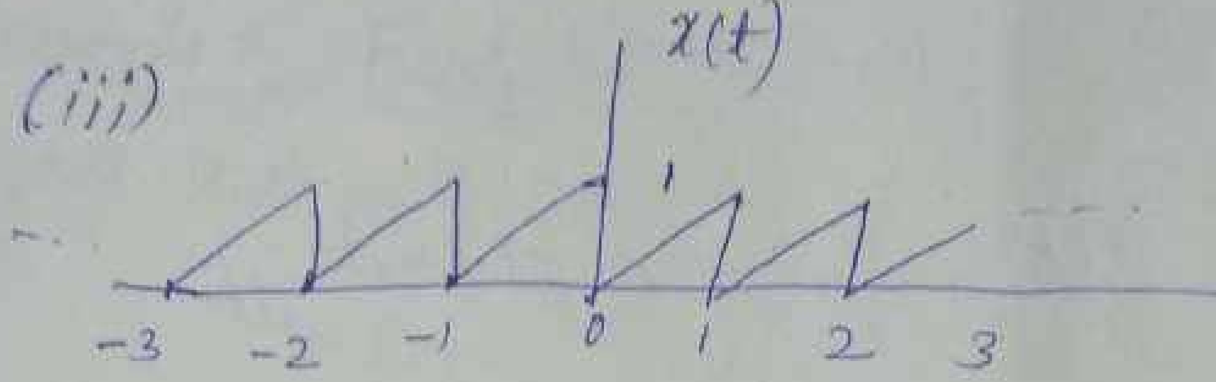
(i)



(ii)



(iii)



3) Find the complex F S coefficients for $x(t)$ given below & plot the Magnitude and phase spectra

$$x(t) = \cos\left(\frac{2\pi}{3}\right)t + 2\cos\left(\frac{5\pi t}{3}\right)$$

Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

$$x(t) \xleftrightarrow{FT} X(\omega)$$

Existance of Fourier transform.

1. On any finite interval.

(a) $x(t)$ is bounded

(b) $x(t)$ has a finite number of maxima and minima

(c) $x(t)$ has a finite number of discontinuities

2. $x(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

Magnitude and Phase Spectra

The Fourier transform $X(\omega)$ in general is a complex quantity and may be expressed in an exponential form as follows

$$X(\omega) = |X(\omega)| \, e^{j\phi(\omega)}$$

plot of $|X(\omega)|$ versus ω — Magnitude spectrum

for real signals $X(-\omega) = X^*(\omega)$

- Non periodic signals have continuous spectra

time domain	F D
Even symmety in $x(t)$	$X(\omega)$ is real and even symmetric
Odd symmety in $x(t)$	$X(\omega)$ is imagi
No symmety in $x(t)$	The real part of $X(\omega)$ is even symmetric and imaginary part of $X(\omega)$ is odd symmetric

Properties of Fourier Transform

Linearity

If $x(t) \longleftrightarrow X(\omega)$

$y(t) \longleftrightarrow Y(\omega)$

then

$$a x(t) + b y(t) \longleftrightarrow a X(\omega) + b Y(\omega)$$

$$F\{a x(t) + b y(t)\} = \int_{-\infty}^{\infty} (a x(t) + b y(t)) e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= a X(\omega) + b Y(\omega).$$

Time shift

$$x(t) \longleftrightarrow X(\omega)$$

$$\underbrace{x(t-t_0)}_{y(t)} \longleftrightarrow \underbrace{e^{-j\omega t_0} X(\omega)}_{Y(\omega)}$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

$t - t_0 = z$ then

$$dt = dz$$

$$Y(\omega) = \int_{-\infty}^{\infty} x(z)\, e^{-j\omega(z+t_0)}\, dz.$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(z)\, e^{-j\omega z}\, dz$$

$$= e^{-j\omega t_0}\, X(\omega)$$

Frequency shift

$$x(t) \longleftrightarrow X(\omega)$$

$$\underbrace{e^{j\beta t}\, x(t)}_{y(t)} \longleftrightarrow X(\omega-\beta)$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} e^{j\beta t}\, x(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\beta)t}\, dt.$$

$$Y(\omega) = X(\omega-\beta)$$

Scaling $x(t) \longleftrightarrow X(\omega)$

$$y(t) = x(at) \longleftrightarrow Y(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

proof

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt \qquad a > 0$$

$$at = z \qquad t = \frac{z}{a}$$

$$dt = \frac{dz}{a}$$

Accordingly $X(\omega) = \int_{-\infty}^{\infty} x(z)\, e^{-j\omega \frac{z}{a}} \, \frac{dz}{a}$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(z)\, e^{-j\omega\left(\frac{z}{a}\right)} dz$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

The scaling property is its own dual such that a scaling by a in time domain results in inverse scaling by $\frac{1}{a}$ and amplitude scaling by $\frac{1}{|a|}$ in frequency domain.

Eg. Compression of $x(t)$ to $x(at)$ leads to expansion of $X(\omega)$ by a and an amplitude reduction by $|a|$. The multiplier $\frac{1}{|a|}$ ensures that the scaled signal in time domain and the scaled spectrum in frequency domain posses the same energy

Frequency differentiation

$$x(t) \longleftrightarrow X(\omega)$$

$$-jt\,x(t) \longleftrightarrow \frac{d}{d\omega} X(\omega)$$

proof

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

d. w. r. t. w.

$$\frac{d}{d\omega} X(\omega) = \int_{-\infty}^{\infty} x(t) \left[-jt\, e^{-j\omega t} \right] dt$$

$$= \int_{-\infty}^{\infty} \left[-jt\, x(t) \right] e^{-j\omega t}\, dt$$

$$-jt\,x(t) \longleftrightarrow \frac{d}{d\omega} X(\omega)$$

Time differentiation

$$x(t) \longleftrightarrow X(\omega)$$

$$\frac{d}{dt} x(t) \longleftrightarrow j\omega X(\omega)$$

Proof

from the definition of IFT, we have.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \longrightarrow ①$$

d.w.r.to t.

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(j\omega\, e^{j\omega t}\right) d\omega$$

$$\frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[j\omega\, X(\omega)\right] e^{j\omega t}\, d\omega \longrightarrow ②$$

(Equating) Comparing equations ① & ②, we get.

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega\, X(\omega)$$

Convolution

if $x(t) \longleftrightarrow X(\omega)$

$y(t) \longleftrightarrow Y(\omega)$

$$\underbrace{x(t) * y(t)}_{z(t)} \longleftrightarrow \underbrace{X(\omega)\,Y(\omega)}_{Z(\omega)}$$

proof

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\, d\tau \int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\, d\tau\; Y(\omega)\, e^{-j\omega\tau}$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau\; Y(\omega)$$

$$Z(\omega) = X(\omega)\, Y(\omega)$$

Modulation.

$$x(t) \longleftrightarrow X(\omega)$$

$$y(t) \longleftrightarrow Y(\omega)$$

$$\underbrace{x(t)\,y(t)}_{z(t)} \longleftrightarrow \underbrace{\frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]}_{Z(\omega)}$$

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty} x(t)\, y(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda$$

substituting in $Z(\omega)$

$$Z(\omega) = \int_{t=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda \right] y(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(\lambda) \int_{t=-\infty}^{\infty} y(t)\, e^{-j(\omega-\lambda)t}\, dt\, d\lambda$$

$$Z(\omega) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(\lambda)\, Y(\omega-\lambda)\, d\lambda$$

$$Z(\omega) = \frac{1}{2\pi} \left[X(\omega) * Y(\omega) \right]$$

Parseval's theorem or Rayleigh's theorem

If $x(t) \longleftrightarrow X(\omega)$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Proof $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$= \int_{t=-\infty}^{\infty} x(t)\, x^*(t)\, dt \longrightarrow ①$$

from the defn of IFT, we have

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

Taking conjugate on both the sides we get

$$x^*(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega.$$

Substituting for $x^*(t)$ in ①.

$$E = \int_{t=-\infty}^{\infty} x(t) \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega\, dt.$$

$$E = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) \left[\int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega)\, X(\omega)\, d\omega$$

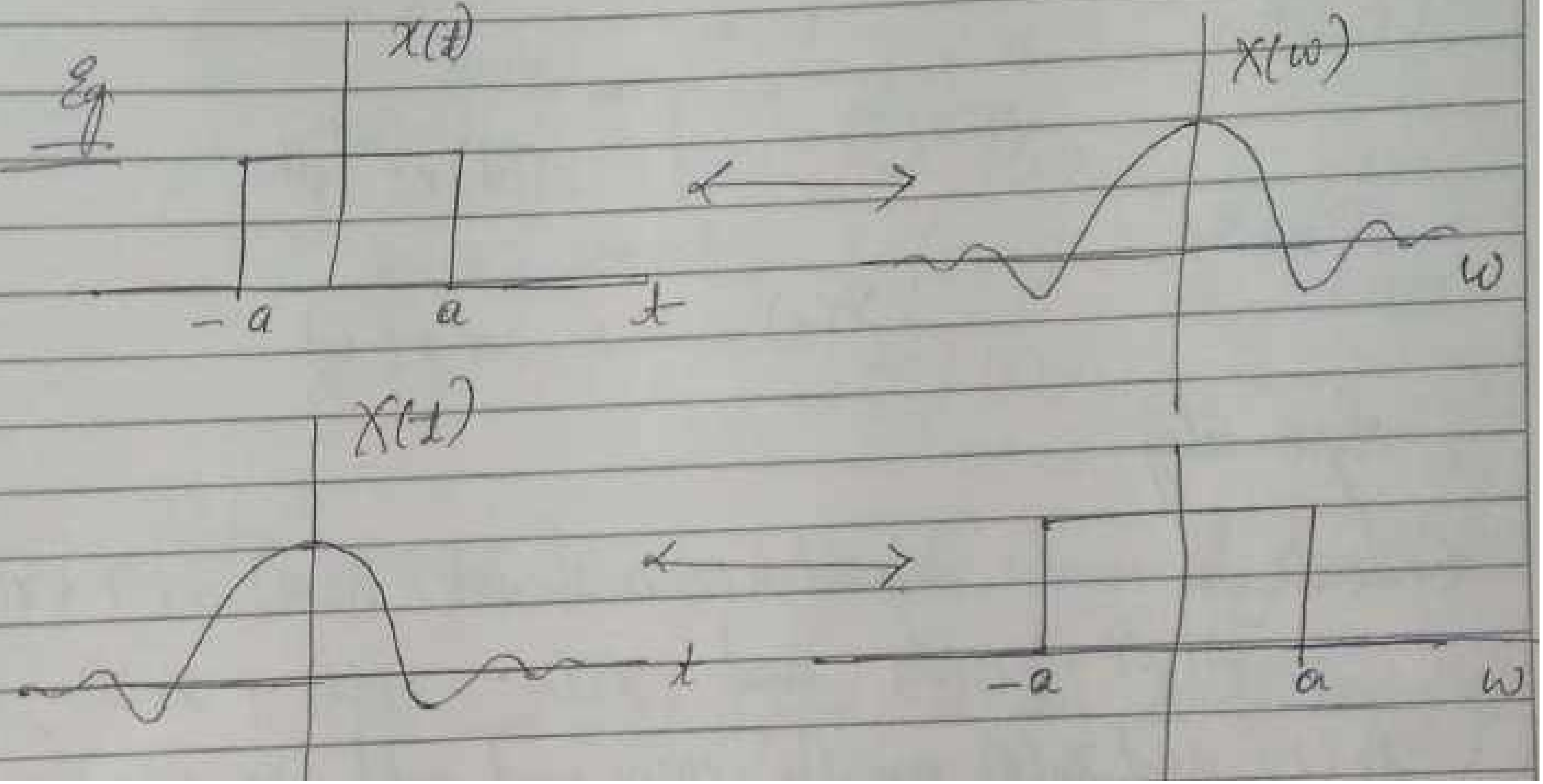
$$\boxed{E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega}$$

Duality or Similarity theorem

if $x(t) \longleftrightarrow X(\omega)$ then

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

Eg



Proof

From the definition of IFT, we have

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

Interchanging t and ω, we get

$$x(\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{\infty} X(t)\, e^{jt\omega}\, dt$$

Replacing ω by $-\omega$ we get

$$x(-\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$2\pi x(-\omega) = \int_{t=-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

Find the fourier transform of the following and sketch its magnitude and phase spectrum.

$x(t) = \delta(t).$

Given $x(t) = \delta(t).$

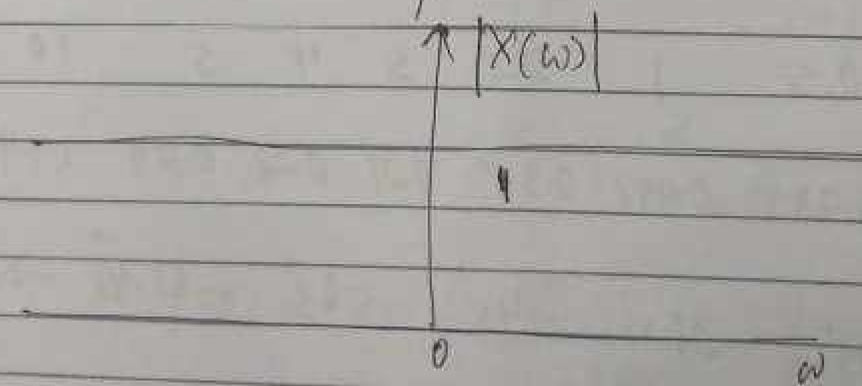
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$
$$= 1 \quad \text{for} \quad t = 0.$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt.$$

$$\boxed{X(\omega) = 1.}$$

$|X(\omega)| = 1$ for all ω



$$x(t) = e^{-at} u(t).$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt.$$

$$= \frac{1}{-(a+j\omega)} \left. e^{-(a+j\omega)t} \right]_{0}^{\infty}$$

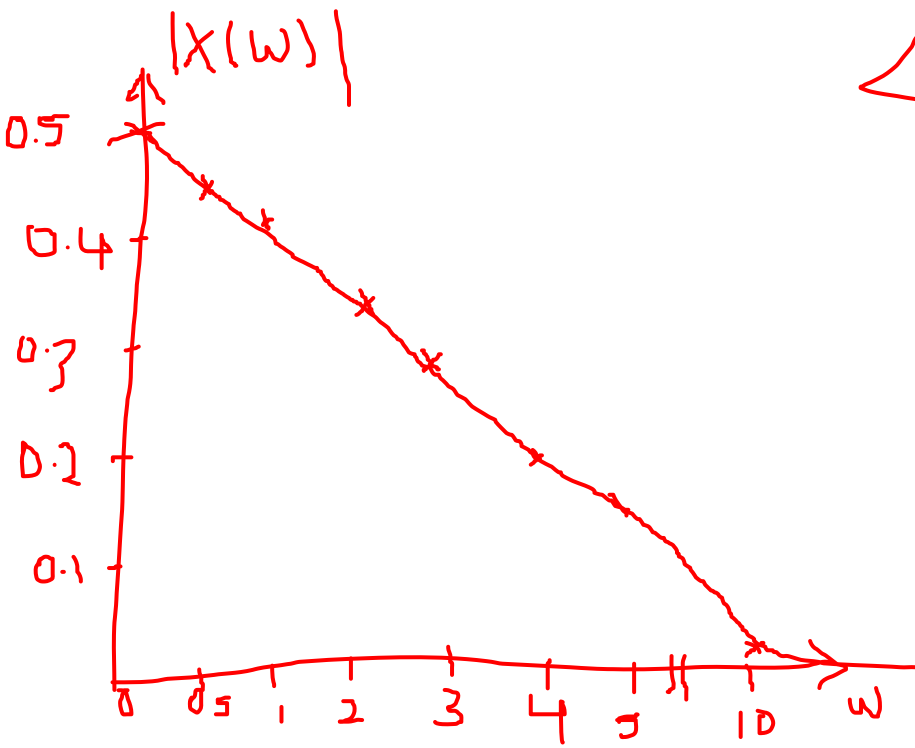
$$X(\omega) = \frac{1}{a+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\angle X(\omega) = -\tan^{-1}(\omega/a)$$

Let us assume $a=2$

ω	0	0.5	1	2	3	4	5	10	∞
$\lvert X(\omega)\rvert$	0.5	0.485	0.447	0.35	0.27	0.22	0.185	0.09	0
$\angle X(\omega)$	0	-14°	-26.5°	-45°	-56.3°	-63.43°	-68°	-78.7°	-90°



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$x(t) = e^{-|t|}$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-t} \cdot e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(1-j\omega)t} dt + \int_{0}^{\infty} e^{-(1+j\omega)t} dt$$

$$= \frac{-1}{(1-j\omega)} e^{-(1-j\omega)t} \Big]_{0}^{\infty} + \frac{-1}{(1+j\omega)} e^{-(1+j\omega)t} \Big]_{0}^{\infty}$$

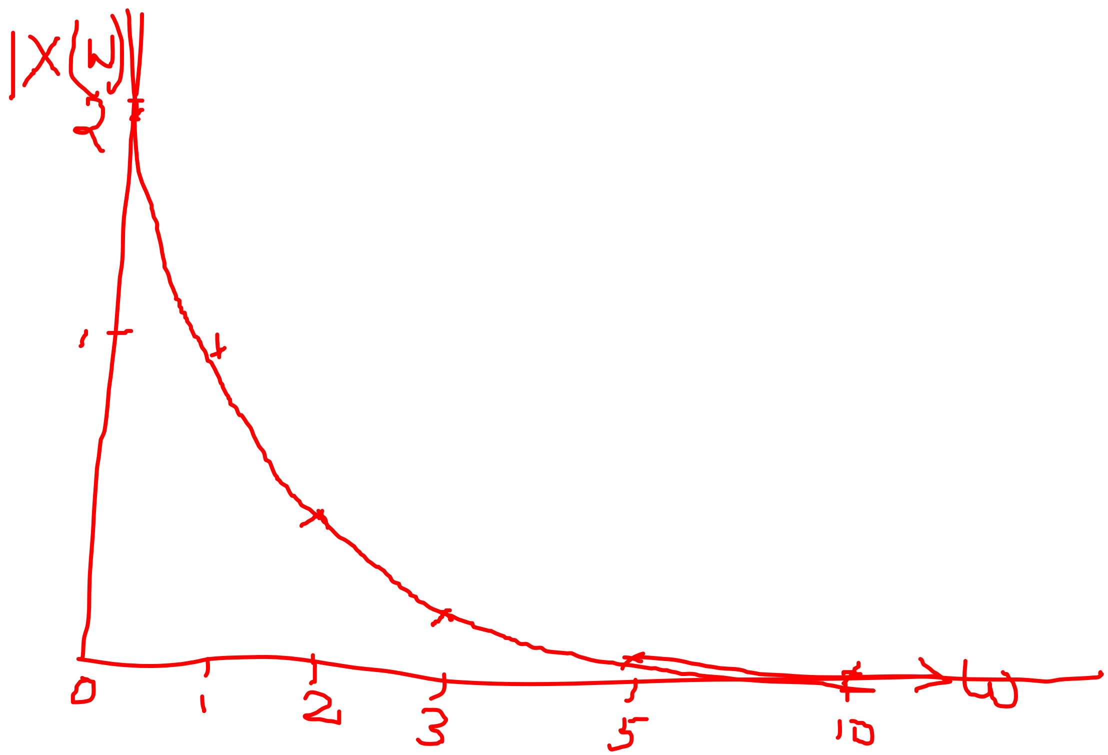
$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega}$$

$$= \frac{2}{1+\omega^2}$$

$$|X(\omega)| = \frac{2}{1+\omega^2} \quad \text{for all } \omega$$

$$\angle X(\omega) = 0 \quad \text{for all } \omega$$

ω (in rads)	0	1	2	3	5	10	∞
$\|X(\omega)\|$	2	1	0.4	0.2	0.0769	0.019	0



(iv) $x(t) = e^{2t} u(t)$

The given signal is not absolutely integrable.

That is $\int_0^{\infty} e^{2t} dt = \infty$. Therefore Fourier transform of $x(t)$ does not exist.

Q) Find the fourier transform of the signal

$$x(t) = \cos(\omega_0 t)$$

Sol:

$$x(t) = \cos \omega_0 t$$

$$= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$X(\omega) = F\left\{\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right\}$$

$$= \frac{1}{2}\left[F\{e^{j\omega_0 t}\} + F\{e^{-j\omega_0 t}\}\right]$$

$$X(\omega) = \frac{1}{2}\left\{2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right\}$$

$$X(\omega) = \pi\left\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right\}$$

$$F(e^{j\omega_0 t}) = 2\pi\delta(\omega-\omega_0)$$

$$e^{j\omega_0 t} = 2\pi F^{-1}\delta(\omega-\omega_0)$$

$$F^{-1}\delta(\omega-\omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,e^{j\omega t}\,dt$$

$$F^{-1}\delta(\omega-\omega_0) = \frac{1}{2\pi}e^{j\omega_0 t}$$

so, $2\pi F^{-1}\delta(\omega-\omega_0) = e^{j\omega_0 t}$

taking FT on BS

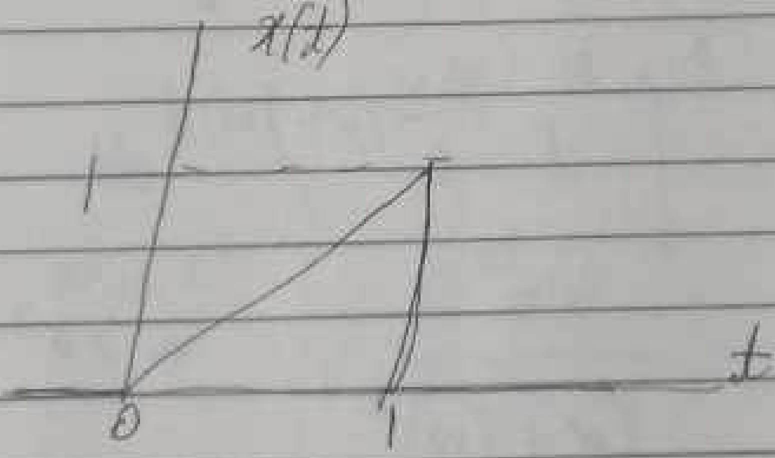
$$F\{2\pi \cancel{F^{-1}}\delta(\omega-\omega_0)\} = F(e^{j\omega_0 t})$$

$$\boxed{F(e^{j\omega_0 t}) = 2\pi\delta(\omega-\omega_0)}$$

Sketch the signal $x(t)$ and find its Fourier transform

$$x(t) = r(t) - r(t-1) - u(t-1)$$

Sln



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{0}^{1} t \cdot e^{-j\omega t} dt$$

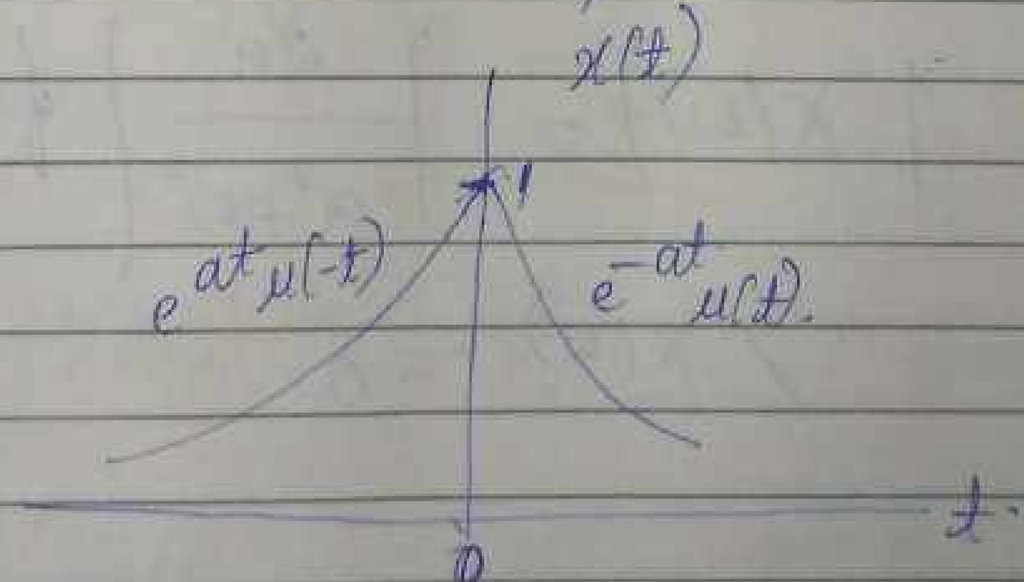
$$X(\omega) = \left. \frac{(1+j\omega t) e^{-j\omega t}}{\omega^2} \right]_0^1$$

$$X(\omega) = \frac{(1+j\omega) e^{-j\omega} - 1}{\omega^2}$$

Find the fourier transform of the signal

$$x(t) = e^{-a|t|} \quad a>0$$

Also, sketch the magnitude and phase spectra.



• As $x(t)$ is a real function, hence $X(-\omega) = X^*(\omega)$.
Hence $|X(\omega)|$ versus ω exhibits even symmetry and $\theta(\omega)$ versus ω exhibits odd symmetry

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

$$|X(\omega)| = \left|\frac{2a}{a^2+\omega^2}\right| \text{ for all } \omega$$

$$\angle X(\omega) = 0$$

Find the Fourier transform of the signum function $x(t) = sgn(t)$. Draw the magnitude & phase spectra.

$$sgn(t) = 1 \qquad t>0$$
$$= -1 \qquad t<0$$

$$x(t) = sgn(t)$$

d. w. r. t. t

$$\frac{d}{dt} x(t) = 2\delta(t)$$

Taking FT on both sides using time differentiation property, we get:

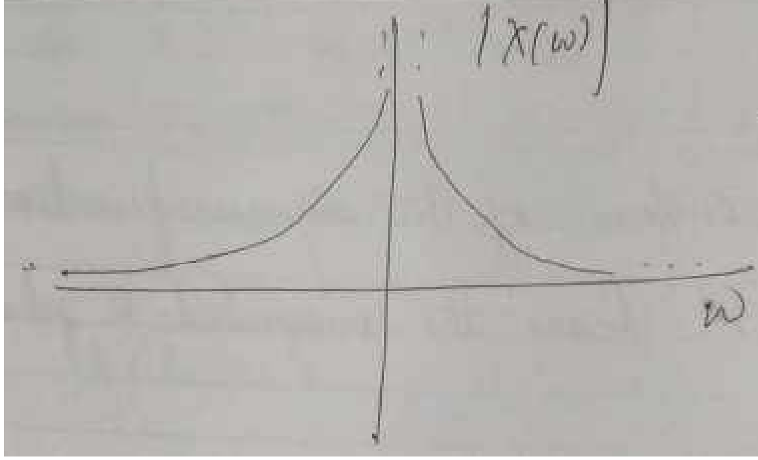
$j\omega \cdot X(\omega) = 2$

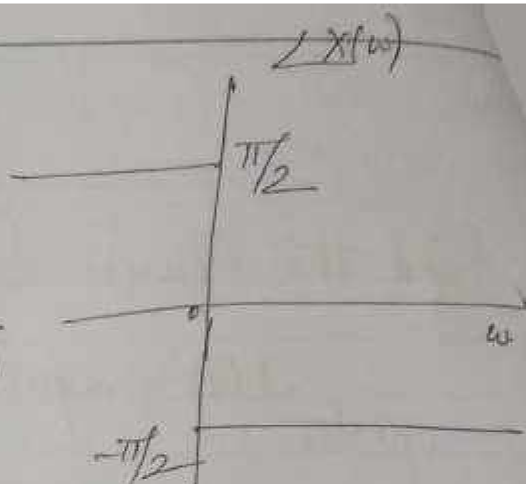
$$\boxed{X(\omega) = \frac{2}{j\omega}}$$

$$|X(\omega)| = \frac{2}{\omega}$$

$$\angle X(\omega) = \frac{-\pi}{2} \qquad \omega > 0$$

$$= \frac{\pi}{2} \qquad \omega < 0$$

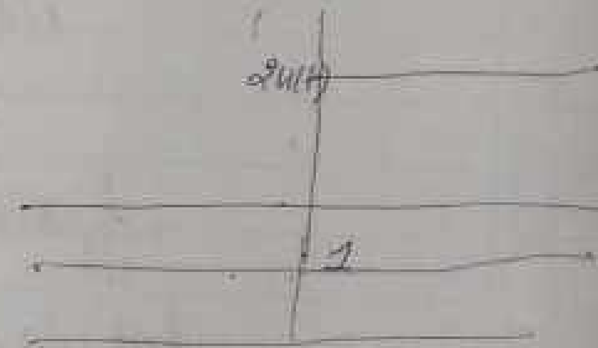




Q) Find the fourier transform of unit step function

$x(t) = u(t)$

W.K.T $\quad sgn(t) = 2u(t) - 1$



$$u(t) = \frac{sgn(t) + 1}{2}$$

$$= \frac{sgn(t)}{2} + \frac{1}{2}$$

Taking FT on both sides

$$F\{u(t)\} = \left(\frac{1}{2}\right) \cdot 2\pi\delta(\omega) + \frac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$\boxed{U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}}$$

Q) find the FT of 1

$x(t) = 1$

Sol^n Here $\int_{-\infty}^{\infty} |x(t)| \, dt = \int_{-\infty}^{\infty} 1 \cdot dt \rightarrow \infty$

$\therefore$ the Dirichlet condition is not satisfied.

$\left[\int_{-\infty}^{\infty} x(t) \, dt \text{ should be absolutely integrable} \right)$.

Still, we can show that FT of $x(t) = 1$ exists by using some of the properties of FT

From duality property, we have,

if $x(t) \longleftrightarrow X(\omega)$, then

$X(t) \longleftrightarrow 2\pi x(\omega)$.

$\therefore$ We have

$\delta(t) \longleftrightarrow 1$.

using duality property, we have

$1 \longleftrightarrow 2\pi\,\delta(-\omega)$

Q) What is the energy of the signal $x(t) = e^{-\alpha t} u(t)$ and what is its energy in the frequency band $|\omega|$ $(-0.5, 0.5)$

Sln

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

$$= \int_{0}^{\infty} e^{-2\alpha t} \, dt = \frac{1}{2\alpha} \text{ Joules.}$$

$$X(\omega) = \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} \, dt. = \int_{0}^{\infty} e^{-(\alpha + j\omega)t} \, dt$$

$$X(\omega) = \frac{1}{\alpha + j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}}$$

The Energy E_B in the band $(-0.5, 0.5)$ is.

$$E_B = \frac{1}{2\pi}\int_{-0.5}^{0.5} |X(\omega)|^2 d\omega$$

$$= \frac{1}{\cancel{2}\pi} \times \cancel{2} \int_{0}^{0.5} \frac{1}{\alpha^2+\omega^2} d\omega$$

$$= \frac{\tan^{-1}\left(\frac{\omega}{\alpha}\right)}{\alpha\pi}\Bigg]_0^{0.5} = \frac{1}{\alpha\pi}\tan^{-1}\left(\frac{0.5}{\alpha}\right) \text{Joules}$$

Discrete time fourier transform.

The DTFT of the signal $x(n)$ is given by

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

DTFT exists only when the infinite summation converges.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega \qquad X(e^{j\Omega})$$

We say that $X(e^{j\Omega})$ and $x(n)$ forms DTFT pair which can be expressed as

$$x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$$

$X(e^{j\Omega})$ is the frequency domain representation of the time domain signal $x(n)$. $X(e^{j\Omega})$ is also known as spectrum of $x(n)$.

Eqn (1) is known as analysis equation and Eqn (2) is synthesis equation

Periodicity:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$X\left(e^{j(\Omega+2k\pi)}\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} \cdot \cancel{e^{-j2\pi k n}}$$

$$X\left(e^{j(\Omega+2k\pi)}\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$X\left(e^{j(\Omega+2k\pi)}\right) = X(e^{j\Omega})$$

This indicates that $X(e^{j\Omega})$ is periodic with period 2π.

properties of DTFT

- → Linearity
- → Time shift
- → frequency shift
- → Scaling
- → frequency differentiation
- → Summation
- → Convolution
- → Modulation
- → Parseval's theorem
- → Symmetry.

Linearity: If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

and

$y(n) \xleftrightarrow{DTFT} Y(e^{j\Omega})$

then $z(n) = ax(n) + by(n) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e$

proof: We have

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}.$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$n=-\infty$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$$\therefore Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[ax(n) + by(n)\right] e^{-j\Omega n}$$

$$= a\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} + b\sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$$\boxed{Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})}$$

Time shift: If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then $\underbrace{x(n-n_0)}_{y(n)} \longleftrightarrow \underbrace{e^{-j\Omega n_0} X(e^{j\Omega})}_{Y(e^{j\Omega})}$.

proof

We have

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-n_0) e^{-j\Omega n}$$

put $m = n - n_0$ then

$$Y(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} x(m) \cdot e^{-j\Omega(m+n_0)}$$

$$Y(e^{j\Omega}) = e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m}$$

$$\boxed{Y(e^{j\Omega}) = e^{-j\Omega n_0} X(e^{j\Omega})}$$

frequency shift

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then

$$\underbrace{e^{j\beta n} \cdot x(n)}_{y(n)} \xleftrightarrow{DTFT} \underbrace{X(e^{j(\Omega-\beta)})}_{Y(e^{j\Omega})}$$

Sanjay

proof We have

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} e^{j\beta n} x(n) \cdot e^{-j\Omega n}$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\Omega-\beta)n}$$

$$\boxed{Y(e^{j\Omega}) = X(e^{j(\Omega-\beta)})}$$

Scaling

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then $z(n) = x(pn) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = X(e^{j\Omega/p})$

Proof: We have

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x(pn)\, e^{-j\Omega n}$$

put $pn = m$ then

$$Z(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(\Omega/p)m}$$

$$\boxed{Z(e^{j\Omega}) = X(e^{j\Omega/p})}$$

frequency differentiation

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then $-jn\,x(n) \xleftrightarrow{DTFT} \frac{d}{d\Omega} X(e^{j\Omega})$

proof We have

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

Differentiate both the sides with respect to Ω, we get.

$$\frac{d}{d\Omega} X(e^{j\Omega}) = \frac{d}{d\Omega}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\right]$$

$$\frac{d}{d\Omega} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{d\Omega} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (-jn) e^{-j\Omega n}$$

$$\frac{d}{d\Omega} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} [-jn x(n)] e^{-j\Omega n}$$

$$\boxed{\frac{d}{d\Omega} X(e^{j\Omega}) \xleftrightarrow{DTFT} -jn x(n)}$$

Convolution

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

$y(n) \xleftrightarrow{DTFT} Y(e^{j\Omega})$

then $z(n) = x(n) * y(n) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = X(e^{j\Omega}) \cdot Y(e^{j\Omega})$

We have.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}.$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\Omega n}$$

$$Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) * y(n)] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} x(l) y(n-l)\right] e^{-j\Omega n}$$

put $n-l=m$, then

$$Z(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} x(l) y(m)\right] e^{-j\Omega(l+m)}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \cdot e^{-j\Omega l} \sum_{m=-\infty}^{\infty} y(m) e^{-j\Omega m}$$

$$Z(e^{j\Omega}) = X(e^{j\Omega}) \cdot Y(e^{j\Omega})$$

Modulation :

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

$y(n) \xleftrightarrow{DTFT} Y(e^{j\Omega})$.

then $z(n) = x(n) \cdot y(n) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = \frac{1}{2\pi}\left[X(e^{j\Omega}) \circledast Y(e^{j\Omega})\right]$

where $\circledast$ indicates periodic convolution.

proof: we have.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\Omega n}$$

$$Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, y(n)\, e^{-j\Omega n}$$

Substituting the expression

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\beta})\, e^{j\beta n}\, d\beta$$ in above Eqn

$$Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\beta})\, e^{j\beta n}\, d\beta\right] e^{-j\Omega n}$$

Inter changing the order of integration & summation, we get

$$Z(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\beta}) \sum_{n=-\infty}^{\infty} y(n)\, e^{j\beta n} e^{-j\Omega n} \cdot d\beta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\beta}) \sum_{n=-\infty}^{\infty} y(n)\, e^{-j(\Omega-\beta)n}\, d\beta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\beta})\, Y(e^{j(\Omega-\beta)})\, d\beta$$

$$= \frac{1}{2\pi}\left[X(e^{j\Omega}) \circledast Y(e^{j\Omega}) \right]$$

Parseval's theorem

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then $$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} \underbrace{|X(e^{j\Omega})|^2}_{\text{energy density spectrum}} d\Omega$$

proof We have $$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{2\pi} X^*(e^{j\Omega})\, e^{-j\Omega n} d\Omega \right]$$

changing the order of summation and integration

$$E = \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\Omega}) \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\Omega}) \cdot X(e^{j\Omega})\, d\Omega$$

$$E = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\Omega})|^2\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\Omega})|^2\, d\Omega$$

Symmetry property.

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then i) $x(n)$ is real

$$x(n) \longleftrightarrow X^*(e^{j\Omega}) = X(e^{-j\Omega})$$

ii) If $x(n)$ is real and even.

$$x(n) \longleftrightarrow Img\{X(e^{j\Omega}) = 0\}.$$

iii) If $x(n)$ is real and odd.

$$x(n) \longleftrightarrow Re\{X(e^{j\Omega})\} = 0.$$

proof since $x(n)$ is real

$$x(n) = x^*(n).$$

Let

$$x(n) = x_e(n) + x_o(n) \longleftrightarrow X(e^{j\Omega}) = X_R(e^{j\Omega}) + jX_I(e^{j\Omega}) \longrightarrow (a)$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega. \longrightarrow (1)$$

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega})\, e^{-j\Omega n}\, d\Omega.$$

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega})\, e^{j(-\Omega n)}\, d\Omega. \longrightarrow (2)$$

Comparing equations (1) and (2) we see that

$$x(n) \longleftrightarrow X^{*}(e^{-j\Omega}) \qquad \left(x(n) = x^{*}(n) \right)$$

also we know that

$$x(n) \longleftrightarrow X(e^{j\Omega})$$

$$\therefore \Longrightarrow \quad X(e^{j\Omega}) = X^{*}(e^{-j\Omega})$$

taking complex conjugate on B.S

$$\boxed{X^{*}(e^{j\Omega}) = X(e^{-j\Omega})}$$

~~We have $x(n) = x_e(n) + x_o(n)$~~

$$x(-n) = x_e(-n) + x_o(-n)$$

$$= x_e(n) - x_o(n)$$

$$\therefore \; x(-n) = x_e(n) - x_o(n) \longleftrightarrow X(e^{-j\Omega}) = X^*(e^{j\Omega})$$

$$= X_R(e^{j\Omega}) - jX_I(e^{j\Omega}) \longrightarrow (b)$$

Adding Eqn (a) and (b)

$$2x_e(n) \longleftrightarrow 2X_R(e^{j\Omega})$$

$$x_e(n) \longleftrightarrow X_R(e^{j\Omega})$$

$\therefore$ DTFT of real and even sequence is purely real.

subtracting (b) from (a), we get

$$2x_o(n) \longleftrightarrow 2jX_I(e^{j\omega})$$

$$x_o(n) \longleftrightarrow jX_I(e^{j\omega}).$$

$\therefore$ DTFT of real and odd sequence is purely imaginary

Q. find the DTFT of the signal

$x(n) = \{1, 3, \underset{\uparrow}{5}, 3, 1\}$ and evaluate $X(e^{j\Omega})$ at $\Omega = 0$

Sol. We have $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

$= x(-2) e^{j2\Omega} + x(-1) e^{j\Omega} + x(0) + x(1) e^{-j\Omega} + x(2) e^{-j2\Omega}$

$= 1 \cdot e^{j2\Omega} + 3e^{j\Omega} + 5 + 3e^{-j\Omega} + 1 \cdot e^{-j2\Omega}$

$\therefore X(e^{j\Omega}) = 5 + 6\cos\Omega + 2\cos 2\Omega$

$X(e^{j\Omega})\Big|_{\Omega=0} = X(e^{j0}) = 13$

Q) Find the DTFT $X(\Omega)$ of the signal $x[n] = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$ and evaluate $X(\Omega)$ at Ω

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-2}^{2} x(n) e^{-j\Omega n}$$

$$= 1 \times e^{j2\Omega} + 2 \times e^{j\Omega} + 3 + 2 \times e^{-j\Omega} + 1 \times e^{-j2\Omega}$$

$$X(\Omega) = 3 + 4\cos\Omega + 2\cos 2\Omega$$

$$X(0) = 3 + 4 + 2 = 9$$

Q). Determine the DTFT of the signal

$x(n) = u(n)$

Solution

we have.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$X(e^{j\Omega}) = \sum_{n=0}^{\infty} 1 \cdot e^{-j\Omega n}$$

$u(n) = 1 \quad n \geq 0$

$u(n) = 0 \quad n < 0$

$$= \sum_{n=0}^{\infty} (e^{-j\Omega})^{n}$$

$$= \frac{1}{1 - e^{-j\Omega}}$$

Q) find the DTFT of the signal

$$x(n) = (-1)^n u(n)$$

Sol: We have

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cancel{u(n)}\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (-1)^n e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \left(-e^{-j\Omega}\right)^n$$

$$= \frac{1}{1 + e^{-j\Omega}}.$$

Q) Find the DTFT of the following signals

(a) $x[n] = (0.5)^{n+2} u[n]$

(b) $x[n] = n(0.5)^{2n} u[n]$

Sol^n

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^{n+2} e^{-j\Omega n}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\Omega}\right)^n$$

$$= \frac{1}{4} \times \frac{1}{1 - \frac{1}{2} e^{-j\Omega}}$$

b)
$$X(\Omega) = \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{2n} e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} n\left(\frac{1}{4} e^{-j\Omega}\right)^{n}$$

$$= \frac{\frac{1}{4} e^{-j\Omega}}{\left(1-\frac{1}{4} e^{-j\Omega}\right)^{2}}$$

$$x[n] = a^{|n|}$$

$$= a^{-n} u(-n-1) + a^{n} u[n]$$

DTFT,
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$\Rightarrow X(\Omega) = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\Omega n}$$

$$= \sum_{n=1}^{\infty} \left(a e^{j\Omega}\right)^n + \sum_{n=0}^{\infty} \left(a e^{-j\Omega}\right)^n$$

$$= \left(\frac{a \cdot e^{j\Omega}}{1 - a e^{j\Omega}}\right) + \left(\frac{1}{1 - a e^{-j\Omega}}\right)$$

$$= \frac{1 - a^2}{1 + a^2 - 2a \cos\Omega}$$

$$x(n) = a^{|n|}$$

$$= a^{-n} u(-n-1) + a^{n} u(n)$$

DTFT,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$\Rightarrow X(\Omega) = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\Omega n}$$

$$= \sum_{n=1}^{\infty} (a e^{j\Omega})^n + \sum_{n=0}^{\infty} (a e^{-j\Omega})^n$$

$$= \left(\frac{a \cdot e^{j\Omega}}{1 - a e^{j\Omega}} \right) + \left(\frac{1}{1 - a e^{-j\Omega}} \right)$$

$$= \frac{1 - a^2}{1 + a^2 - 2a \cos\Omega}$$

Q) $x(n) = \delta(6-3n)$. Also plot the magnitude and phase spectra.

$$x(n) = \delta(6-3n)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(6-3n)\, e^{-j\Omega n}$$

$$= e^{-j\Omega n}\Big|_{6-3n=0}$$

$$= e^{-j\Omega n}\Big|_{n=2}$$

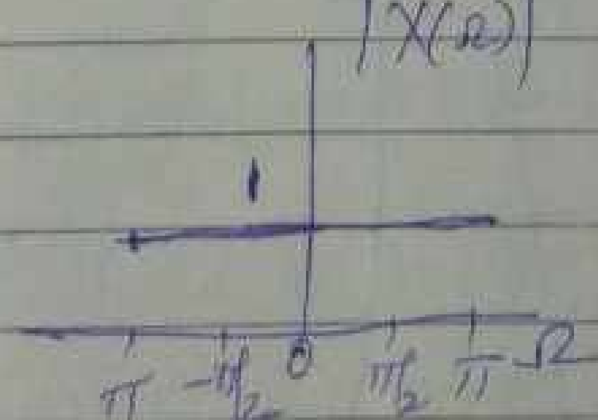
$X(\Omega) = e^{-j2\Omega}$

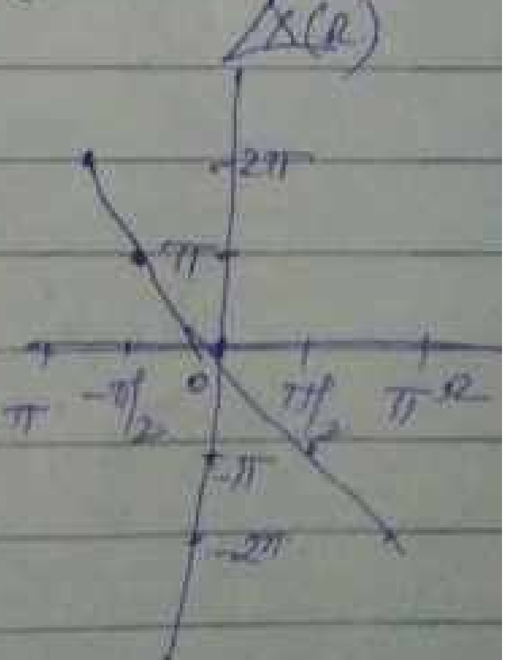
$e^{\pm j\theta} = 1\angle \pm\theta$

Mag spectrum $|X(\Omega)| = 1$

phase spectrum $\theta(\Omega) = \angle X(\Omega) = -2\Omega$

Ω	$\theta(\Omega)$
$-\pi$	2π
$-\pi/2$	π
0	0
$\pi/2$	$-\pi$
π	-2π





Q/

$$x(n) = (\alpha^n \sin \Omega_0 n)\, u(n)$$

Sol:- $X(e^{j\Omega}) = \sum_{n=0}^{\infty} \alpha^n \sin \Omega_0 n \; e^{-j\Omega n}$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \alpha^n \left[e^{j\Omega_0 n} - e^{-j\Omega_0 n} \right] e^{-j\Omega n}$$

$$= \frac{1}{2j} \left\{ \sum_{n=0}^{\infty} \alpha^n e^{j\Omega_0 n} \cdot e^{-j\Omega n} - \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega_0 n} e^{-j\Omega n} \right\}$$

$$= \frac{1}{2j} \left\{ \sum_{n=0}^{\infty} \alpha^n \cdot e^{-j(\Omega - \Omega_0)n} - \sum_{n=0}^{\infty} \alpha^n e^{-j(\Omega + \Omega_0)n} \right\}$$

$$X(e^{j\Omega}) = \frac{1}{2j}\left\{ \frac{1}{1-\alpha e^{-j(\Omega-\Omega_0)}} - \frac{1}{1-\alpha e^{-j(\Omega+\Omega_0)}} \right\}$$

$$= \frac{1}{2j}\left\{ \frac{\left(1-\alpha e^{-j(\Omega+\Omega_0)}\right) - \left(1-\alpha e^{-j(\Omega-\Omega_0)}\right)}{\left(1-\alpha e^{-j(\Omega-\Omega_0)}\right)\left(1-\alpha e^{-j(\Omega+\Omega_0)}\right)} \right\}$$

$$X(e^{j\Omega}) = \frac{\alpha \sin\Omega_0 \cdot e^{-j\Omega}}{1 - 2\alpha\cos\Omega_0 \cdot e^{-j\Omega} + \alpha^2 e^{-j2\Omega}}$$

Q. $x(n) = u(n+1) - u(n-2)$

sketch the spectrum $X(\Omega)$ over

$-\pi \leq \Omega \leq \pi$

Sol

$x(n) = u(n+1) - u(n-2)$

$= \delta(n+1) + \delta(n) + \delta(n-1)$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

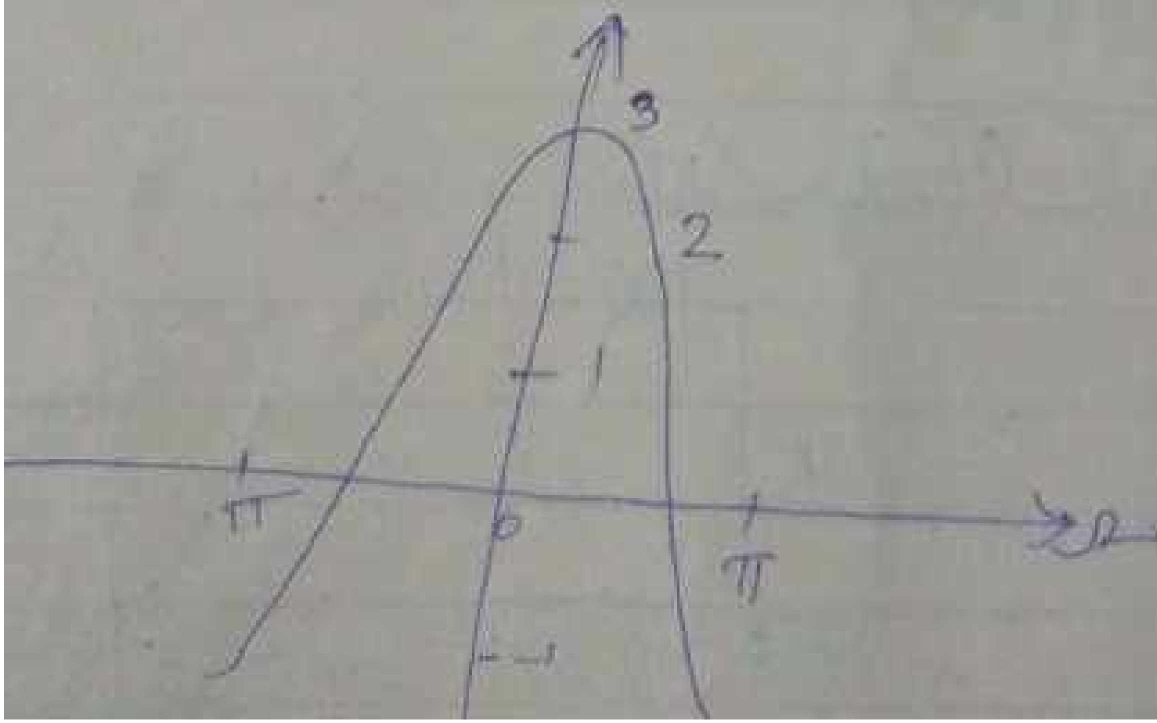
$$= \sum_{n=-\infty}^{\infty} \left[\delta(n+1) + \delta(n) + \delta(n-1)\right] e^{-j\Omega n}$$

$$= e^{-j\Omega n}\Big|_{n=-1} + e^{-j\Omega n}\Big|_{n=0} + e^{-j\Omega n}\Big|_{n=1}$$

$$= e^{-j\Omega n}\Big|_{n=-1} + e^{-j\Omega n}\Big|_{n=0} + e^{-j\Omega n}\Big|_{n=1}$$

$$= e^{j\Omega} + 1 + e^{-j\Omega}$$

$$= 1 + 2\cos\Omega$$



Q) The DTFT of a real signal $x(n)$ is $X(\Omega)$. How is the DTFT of the following signals is related to $X(\Omega)$?

(a) $y(n) = x(-n)$

(b) $g(n) = x(n) * x(-n)$

(c) $s(n) = (-1)^n x(n)$

(d) $z(n) = [1 + \cos n\pi] x(n)$

(e) $b(n) = (-1)^{n/2} x(n)$

Sol $z($

(a) $y(n) = x(-n)$

$Y(\Omega) = X(-\Omega)$.

(b) $g(n) = x(n) * x(-\Omega)$

$G(\Omega) = X(\Omega) X(-\Omega) = |X(\Omega)|^2$.

(c) $s(n) = (-1)^n x(n)$

$$= e^{j\pi n} x(n)$$

$$S(\Omega) = X(\Omega - \pi)$$

(d) $z(n) = (1 + \cos n\pi) x(n)$

$$= \left[1 + (-1)^n\right] x(n)$$

$$= x(n) + e^{j\pi n} x(n)$$

$$Z(\Omega) = X(\Omega) + X(\Omega - \pi)$$

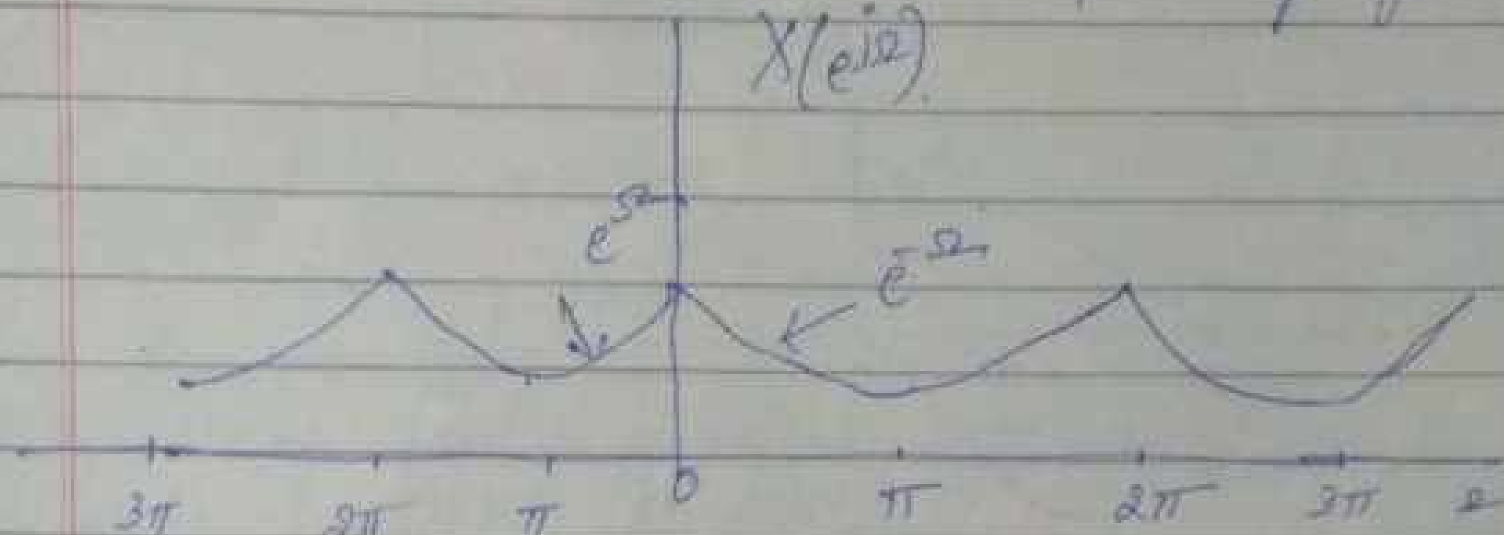
c) $b(n) = (-1)^{\frac{n}{2}} x(n)$

$$= j^{n} x(n)$$

$$= e^{j\frac{n\pi}{2}} x(n)$$

$$B(\Omega) = X\left(\Omega - \frac{\pi}{2}\right)$$

Q) Find the time-domain signal corresponding to the DTFT shown in the following figure!



Sol

$$X(e^{j\Omega}) = e^{\Omega} \qquad -\pi < \Omega < 0$$

$$= e^{-\Omega} \qquad 0 < \Omega < \pi$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{\Omega} e^{j\Omega n}\, d\Omega + \int_{0}^{\pi} e^{-\Omega} e^{j\Omega n}\, d\Omega\right]$$

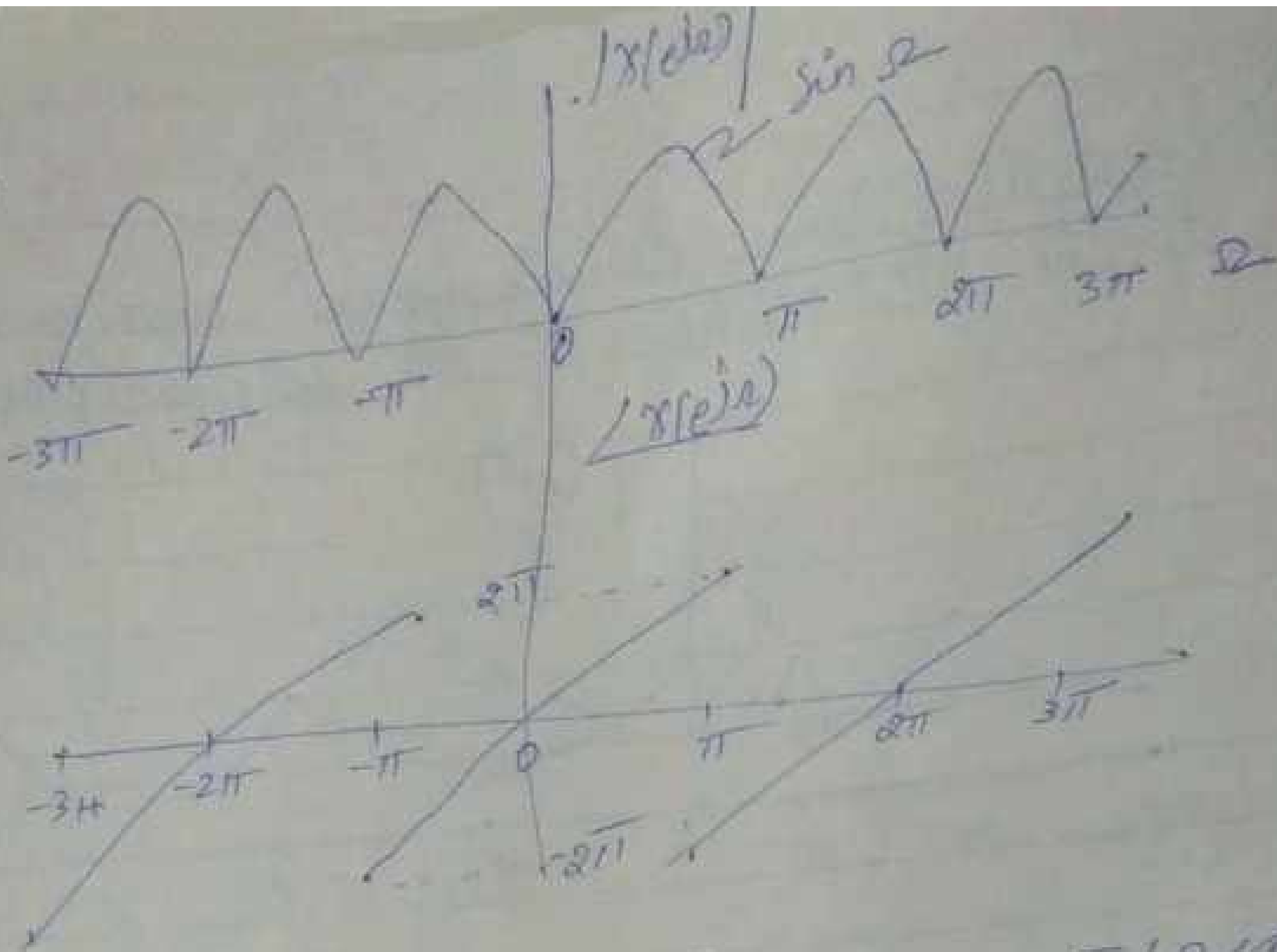
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{(j\Omega+1)\Omega}\, d\Omega + \int_{0}^{\pi} e^{(jn-1)\Omega}\, d\Omega\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{(jn+1)\Omega}}{(jn+1)}\right]_{-\pi}^{0} + \left[\frac{e^{(jn-1)\Omega}}{(jn-1)}\right]_{0}^{\pi}\right\}$$

$$x(n) = \frac{1+(-1)^n}{\pi(n^2+1)}$$

Monika's

Q.



Sol^n

$$X(\Omega) = -\sin\Omega \, e^{j2\Omega} \qquad -\pi < \Omega < 0$$

$$= \sin\Omega \, e^{j2\Omega} \qquad 0 < \Omega < \pi$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{0} (-\sin\omega) e^{j2\omega} e^{j\omega n} d\omega + \int_{0}^{\pi} \sin\omega \, e^{j2\omega} e^{j\omega n} d\omega \right\}$$

$$= \frac{1}{2\pi} \left\{ -\int_{-\pi}^{0} \frac{e^{j(n+3)\omega} - e^{j(n+1)\omega}}{2j} d\omega + \int_{0}^{\pi} \frac{e^{j(n+3)\omega} - e^{j(n+1)\omega}}{2j} d\omega \right\}$$

$$x[n] = \left[1-(-1)^{n+1}\right]\left[\frac{-1}{\pi(n+1)(n+3)}\right] \quad ; \quad n \neq -1, -3$$

and $x(-1) = 0$

$x(-3) = 0$.

Q) Using partial fraction expansion determine the inverse DTFT of the signal.

$$X(\Omega) = \frac{3 - \frac{1}{4}e^{-j\Omega}}{-\frac{1}{16}e^{-j2\Omega} + 1}$$

$$X(\Omega) = \frac{3 - \frac{1}{4}e^{-j\Omega}}{\left(1 - \frac{1}{4}e^{-j\Omega}\right)\left(1 + \frac{1}{4}e^{-j\Omega}\right)}$$

using partial fraction expansion

$$X(\Omega) = \frac{1}{\left(1 - \frac{1}{4}e^{-j\Omega}\right)} + \frac{2}{\left(1 + \frac{1}{4}e^{-j\Omega}\right)}$$

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1 - a e^{-j\Omega}}$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(-\frac{1}{4}\right)^n u[n]$$

$$x[n] = \left\{\left(\frac{1}{4}\right)^n + 2\left(-\frac{1}{4}\right)^n\right\} u[n]$$

Q) find the inverse DTFT of

$$X(e^{j\Omega}) = \frac{6}{e^{-j2\Omega} - 5e^{-j\Omega} + 6}$$

$$= \frac{6}{(e^{-j\Omega} - 2)(e^{-j\Omega} - 3)}$$

Using partial fraction expansion.

$$X(e^{j\Omega}) = \frac{-6}{(e^{-j\Omega}-2)} + \frac{6}{(e^{-j\Omega}-3)}$$

$$= \frac{3}{(1-\frac{1}{2}e^{-j\Omega})} + \frac{(-2)}{(1-\frac{1}{3}e^{-j\Omega})}$$

$$x(n) = 3\left(\tfrac{1}{2}\right)^n u(n) - 2\left(\tfrac{1}{3}\right)^n u(n)$$

Q) find the inverse DTFT of the following

(i) $X(\Omega) = 1 + 2\cos\Omega + 3\cos 2\Omega$

$= 1 + 2\left[\frac{e^{j\Omega} + e^{-j\Omega}}{2}\right] + 3\left[\frac{e^{j2\Omega} + e^{-j2\Omega}}{2}\right]$

$= 1 + e^{j\Omega} + e^{-j\Omega} + \frac{3}{2} e^{j2\Omega} + \frac{3}{2} e^{-j2\Omega}$

$\delta[n] \longleftrightarrow 1$

$\delta[n-n_0] \longleftrightarrow e^{-j\Omega n_0}$

$\delta[n+n_0] \longleftrightarrow e^{+j\Omega n_0}$

taking I-DTFT.

$$x[n] = \delta[n] + \delta[n-1] + \delta[n+1] + \frac{3}{2}\delta[n+2] + \frac{3}{2}\delta[n-2]$$

$$\therefore x[n] = \left\{ \frac{3}{2}, 1, \underset{\uparrow}{1}, 1, \frac{3}{2} \right\}$$

Z-transform

Z-transform is the discrete-time counterpart to the Laplace transform.

Representing signals using discrete time complex exponential.

- Used to analyse signals and systems / system characteristics
  Implement LTI systems / discrete-time systems on computer

Z-transform of a discrete-time signal is given by

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

where z is a complex variable.

$$z = r \cdot e^{j\Omega}$$

$$\left.\begin{array}{l} r - \text{magnitude} \\ \Omega - \text{angle} \end{array}\right\} \text{of } z$$

Substituting for z, we get

$$X(r \cdot e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \left(r \cdot e^{j\Omega}\right)^{-n}$$

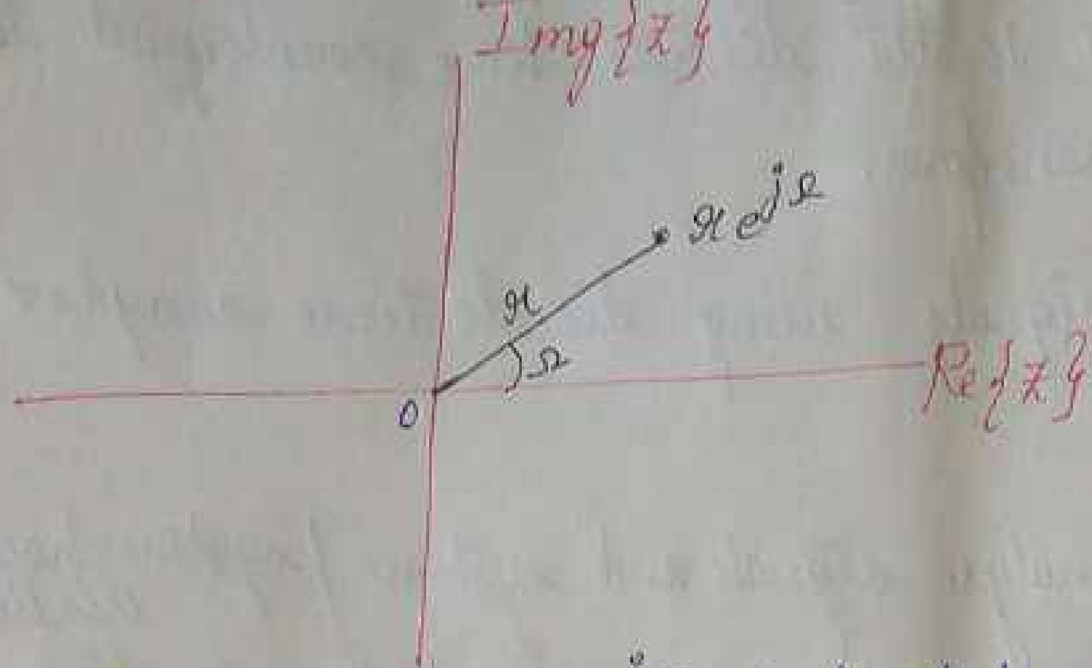
$$X(r \cdot e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left\{ x(n)\, r^{-n} \right\} e^{-j\Omega n}$$

when $r = 1$

$$X(e^{j\Omega}) = X(z)\Big|_{z=e^{j\Omega}}$$

DTFT

Z-plane



A point $z = re^{j\Omega}$ is located at a distance r from the origin and angle Ω relative to the real axis.

The relationship between $x(n)$ and $X(z)$ is given by

$$x(n) \xleftrightarrow{z} X(z)$$

For the existance of $X(z)$, the summation should converge.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{x(n) \cdot r^{-n}\}\, e^{-j\omega n}$$

$$\sum_{n=-\infty}^{\infty} |x(n) \cdot r^{-n}| < \infty .$$

The range of 'r' for which this condition is satisfied is known as Region of Convergence (ROC)

or

The set of $|z|$ for which the summation $X(z)$ convergence is known as ROC.

Example : find the z-transform of the sequences

(1) $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n}$$

$\delta[n] = 1 \quad n = 0$
$\quad\;\; = 0 \quad n \neq 0$

$$X(z) = 1 \cdot z^{0} = 1$$

Region of convergence is the entire Z-plane

(ii) $x[n] = \delta[n-k]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n-k]\, z^{-n}$$

$\delta[n-k] = 1 \quad n=k$

$$= 1 \cdot z^{-k}$$

$$X(z) = \frac{1}{z^{k}}$$

ROC is the entire z-plane, except $z=0$

(iii) $x[n] = \delta[n+k]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+k]\, z^{-n}$$

$\delta[n+k] = 1$

$n = -k$

$$= 1 \cdot z^{-(-k)}$$

$$= z^{k}$$

ROC is the entire z-plane except $z = \infty$

(iv) $x[n] = \{1, 2, 0, 7\}$

$$X(z) = \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 1 + 2 \times z^{-1} + 0 \times z^{-2} + 7 \times z^{-3}$$

$$X(z) = 1 + \frac{2}{z} + \frac{7}{z^3}$$

ROC is the entire z-plane except $z = 0$

(v) $x[n] = \{ 2, -7, 4, \underset{\uparrow}{6}, 1 \}$

$n=-3 \quad n=-2 \quad n=-1 \quad n=0 \quad n=1$

$$X(z) = \sum_{n=-3}^{1} x[n] z^{n}$$

$$= x[-3] z^{3} + x[-2] z^{2} + x[-1] z + x[0] + x(1) z^{-1}$$

$$X(z) = 2z^{3} - 7z^{2} + 4z + 6 + \frac{1}{z}$$

ROC is the entire z-plane except $z = 0$ & $z = \infty$

Properties of Region of Convergence

1. ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

2. The ROC doesnot contain any poles

3. If $x[n]$ is of finite duration, then the ROC is the entire z-plane except $z=0$ and or $z=\infty$

4. If $x[n]$ is a right sided sequence, then the ROC is outside the circle.

5. If $x[n]$ is a left sided sequence, the ROC is inside the circle.

6. If $x[n]$ is a two sided sequence, then the ROC is the concentric ring

7. If the Z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded.

Example: Find the z-transform of the following sequences and plot its ROC.

1. Given. $x[n] = u[n]$

Sol^n: We have $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$$= \sum_{n=-\infty}^{\infty} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \frac{1}{1-z^{-1}}$$

$$X(z) = \frac{z}{z-1}$$

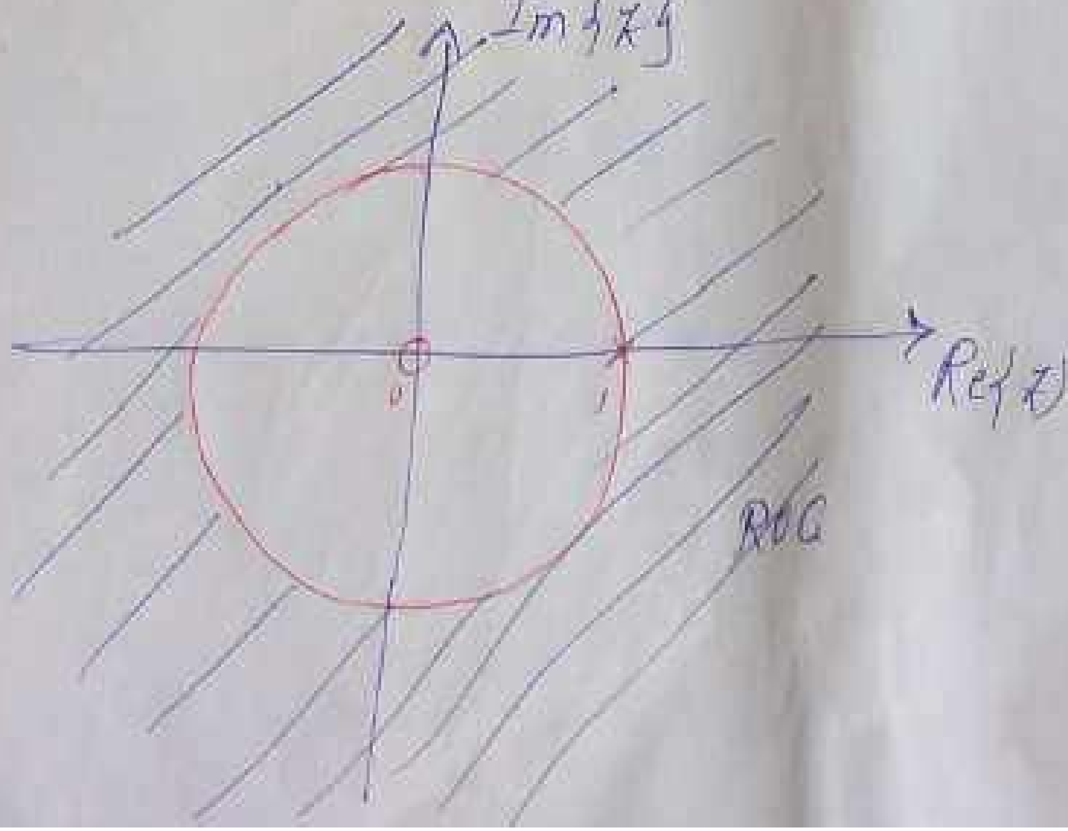
$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$|a| < 1$

$X(z)$ will converge if $|z^{-1}| < 1$

$\therefore$ ROC $|z| > 1$

The ROC and pole-zeros are as shown



2. $x[n] = \left(\frac{1}{2}\right)^n u[n-2]$

Solⁿ

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n-2] \cdot z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha}$$

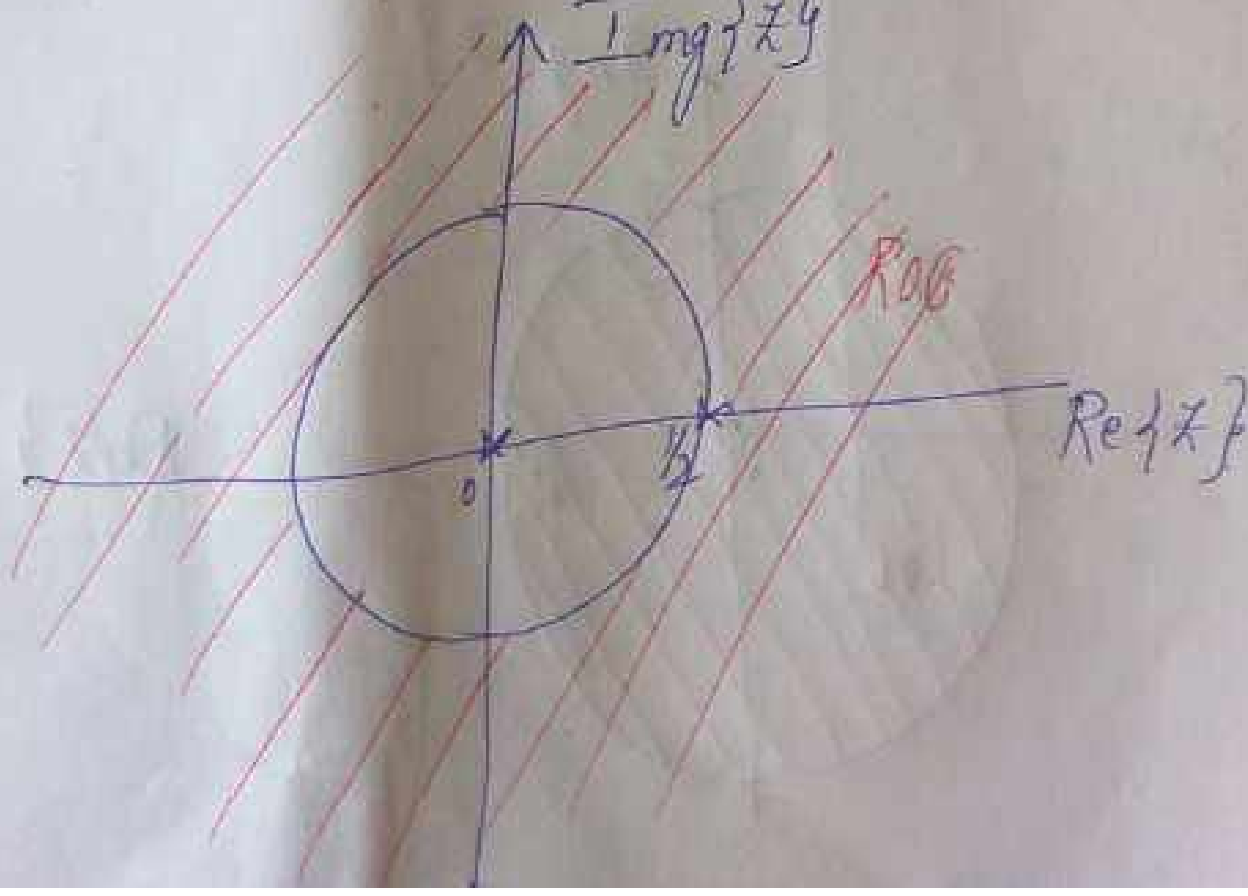
if $|\alpha| < 1$

$$X(z) = \frac{\left(\frac{1}{2} z^{-1}\right)^2}{1 - \frac{1}{2} z^{-1}}$$

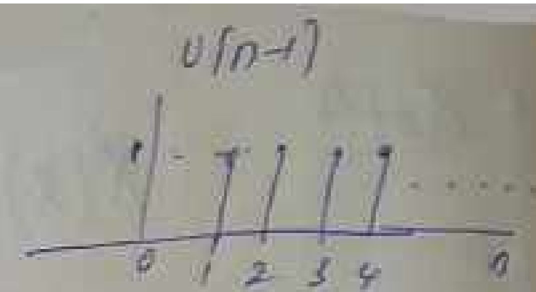
$$= \frac{\frac{1}{4} z^{-2}}{1 - \frac{1}{2} z^{-1}} = \frac{1}{4z\left(z - \frac{1}{2}\right)}$$

$X(z)$ will converge if $\left|\frac{1}{2} z^{-1}\right| < 1$

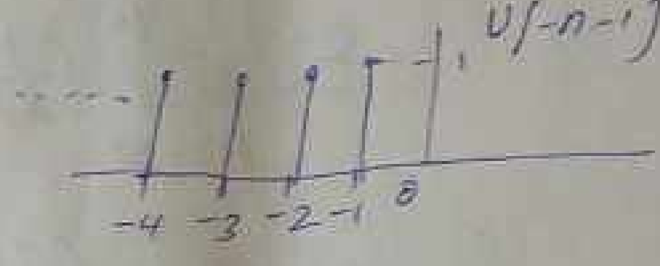
i.e. $|z| > \frac{1}{2}$



3) $x[n] = 2^{n} u[-n-1]$



Sol^n $X(z) = \sum_{n=-\infty}^{-1} 2^{n} z^{-n}$



$$= \sum_{n=-\infty}^{-1} (2z^{-1})$$

$$= \sum_{n=1}^{\infty} (2^{-1}z)^{n}$$

$$\sum_{n=1}^{\infty} a^{n} = \frac{a}{1-a}$$

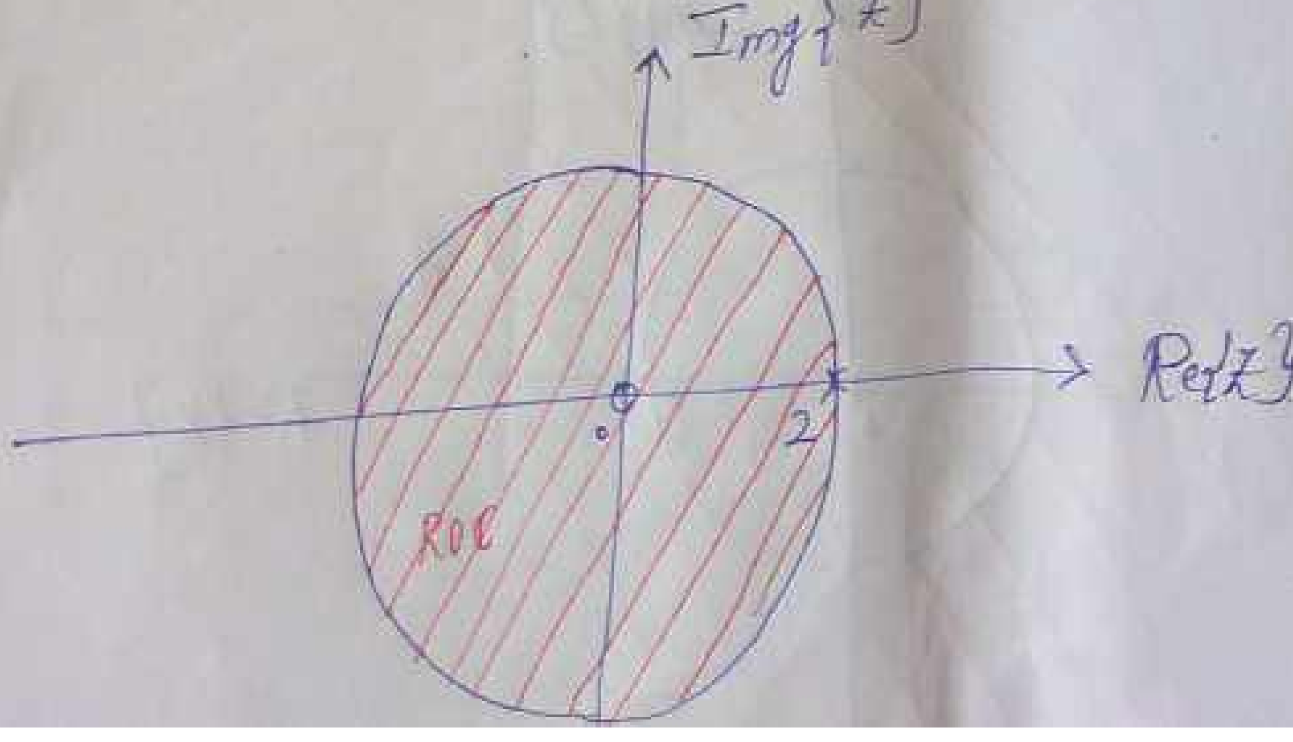
$|a| < 1$

$$= \frac{2^{-1}z}{1-2^{-1}z} = \frac{1/2\, z}{1-\frac{1}{2}z}$$

$$X(z) = \frac{-z}{z-2}$$

$X(z)$ will converge if $|2^{-1}z| < 1$

i.e. $|z| < 2$



47 $x[n] = 3\left(-\frac{1}{2}\right)^n u[n] - 2\left[3^n u[-n-1]\right]$

Sol^n $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} 3\left(-\tfrac{1}{2}\right)^n u[n]\, z^{-n} + \sum_{n=-\infty}^{\infty} \{-2\, 3^n u[-n-1]\} z^{-n}$$

$$= 3\sum_{n=0}^{\infty} \left(-\tfrac{1}{2} z^{-1}\right)^n + (-2)\sum_{n=-\infty}^{-1} \left(3 z^{-1}\right)^n$$

$$= 3\,\frac{1}{1-\left(\tfrac{1}{2} z^{-1}\right)} + (-2)\sum_{n=1}^{\infty} \left(\tfrac{1}{3} z\right)^n$$

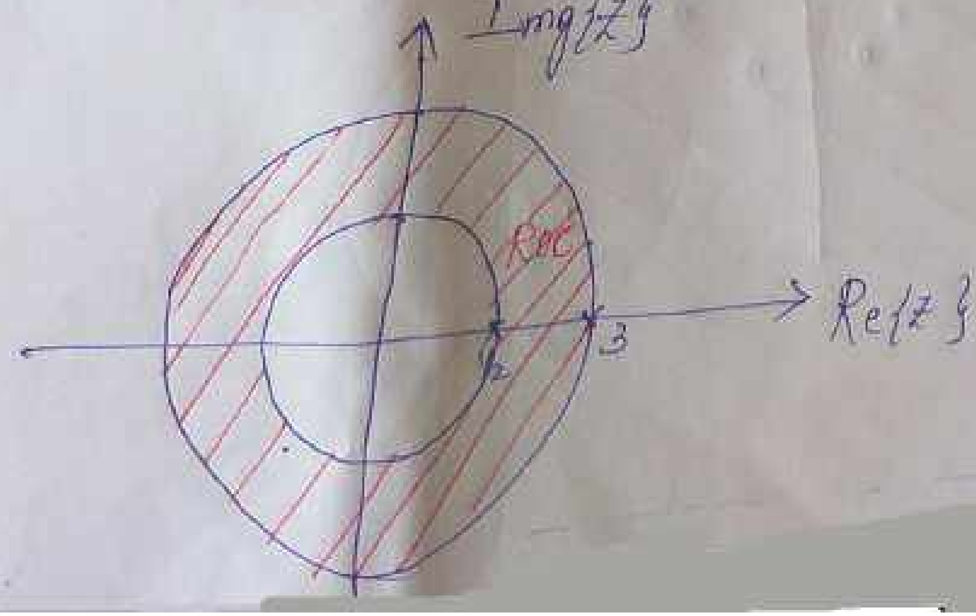
$$= \frac{3z}{z+\tfrac{1}{2}} + (-2)\,\frac{\tfrac{1}{3} z}{1-\tfrac{1}{3} z}$$

$$X(z) = 3 \cdot \frac{z}{z+\frac{1}{2}} + 2\frac{z}{z-3}$$

$$X(z) = \frac{3z(z-3) + 2z(z+\frac{1}{2})}{(z+\frac{1}{2})(z-3)}$$

$X(z)$ will converge $|\frac{1}{2}z^{-1}| < 1$ $\quad |z| > \frac{1}{2}$

$|\frac{1}{3}z| < 1$ $\quad |z| < 3$



5) $x[n] = \left(\frac{1}{2}\right)^n \left\{ u[n] - u[n-10] \right\}$

Sol^n.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \underbrace{\left\{ u[n] - u[n-10] \right\}}_{1} \cdot z^{-n}$$

$$= \sum_{n=0}^{9} \left(\frac{1}{2} z^{-1}\right)^n$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} \; ; \text{if } \alpha \neq 1$$

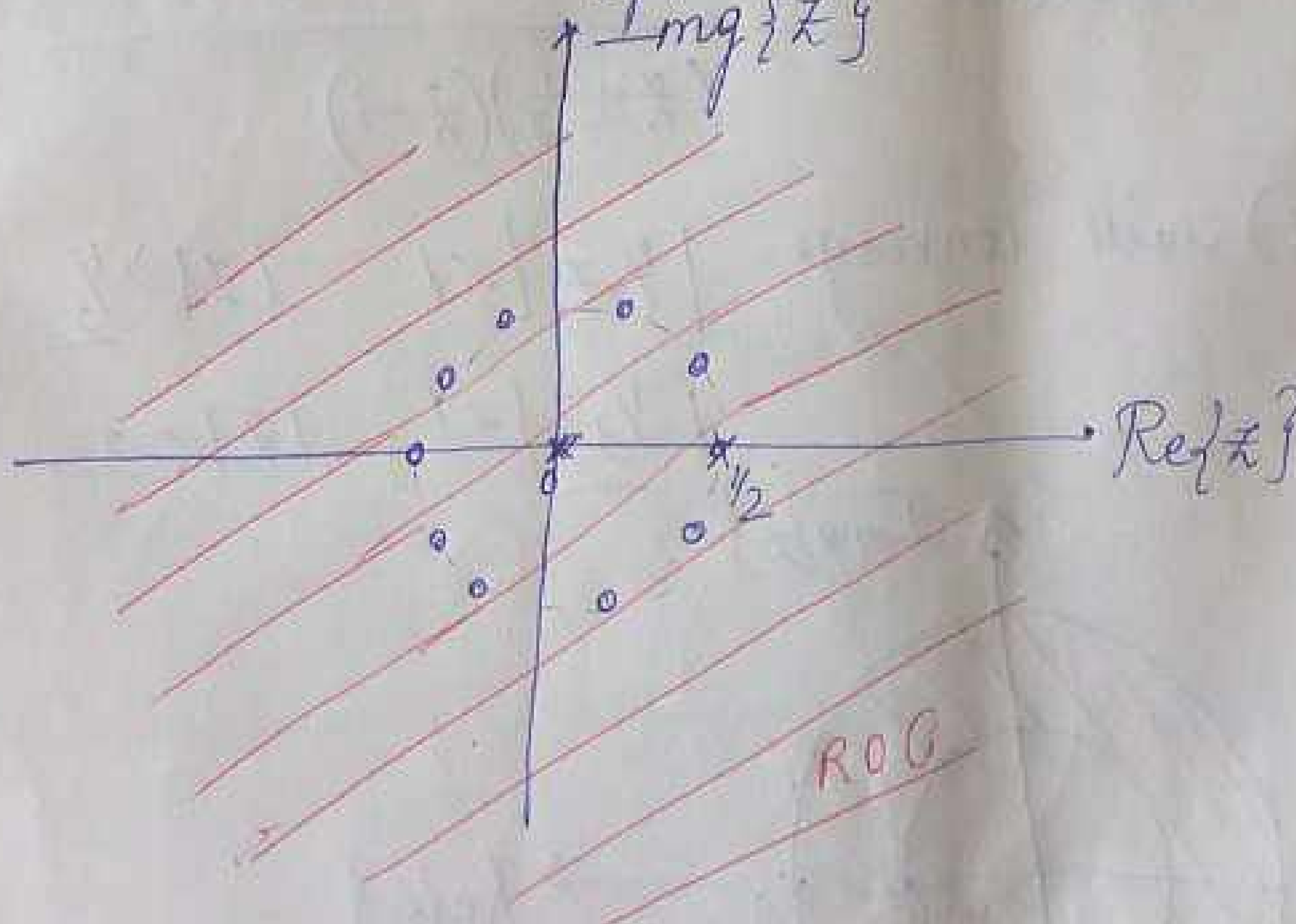
$$= N \text{ if } \alpha = 1$$

$$X(z) = \frac{1 - \left(\frac{1}{2} z^{-1}\right)^{10}}{1 - \frac{1}{2} z^{-1}}$$

$$X(z) = \frac{1}{z^9} \left[\frac{z^{10} - \left(\frac{1}{2}\right)^{10}}{z - \frac{1}{2}} \right]$$

As it is a finite sequence.

ROC is Entire Z-plane except Z=0.



6) $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

Sol^n $X(z) = \sum_{n=-\infty}^{\infty}\left[7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]\right] z^{-n}$

$$= \sum_{n=0}^{\infty}\left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right] z^{-n}$$

$$= 7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n - 6\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

$$= 7 \cdot \frac{1}{1-\frac{1}{3}z^{-1}} - 6\frac{1}{1-\frac{1}{2}z^{-1}}$$

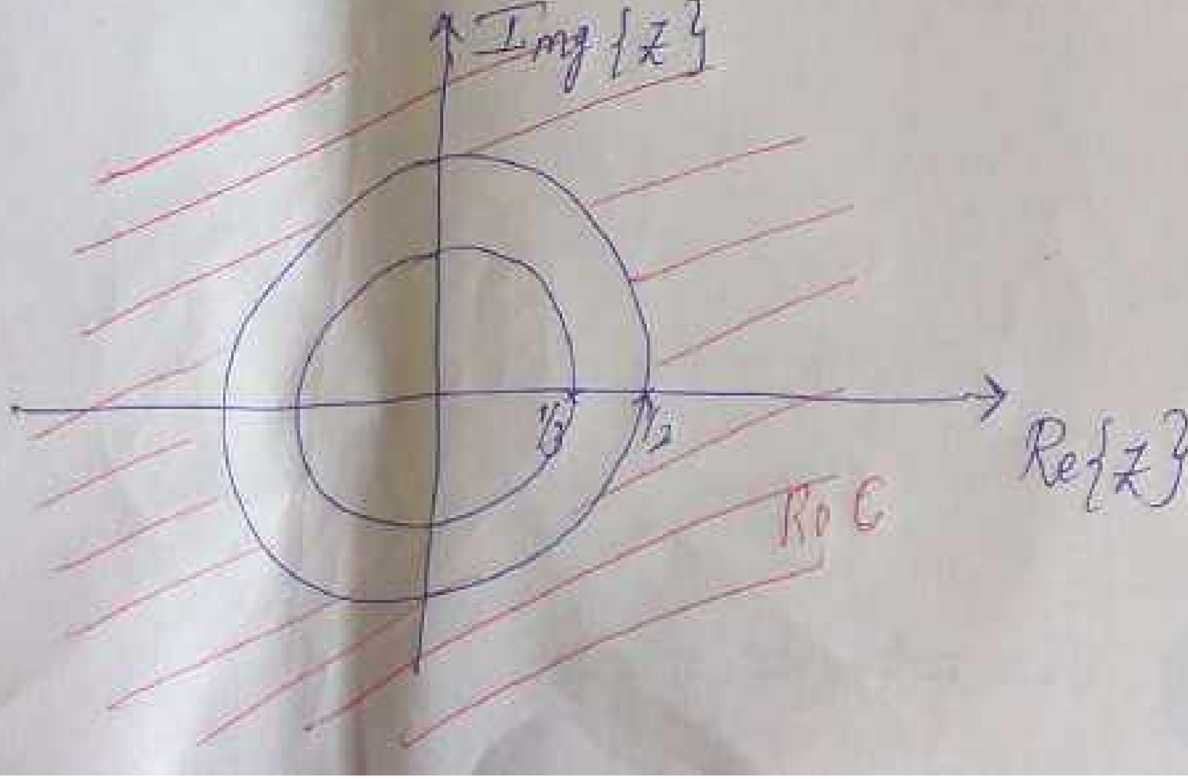
$$= 7\left(\frac{z}{z-1/3}\right) - 6\left(\frac{z}{z-\frac{1}{2}}\right) = \frac{7z(z-1/2) - 6z(z-1/3)}{(z-1/3)(z-1/2)}$$

X(z) will converge if

$$\left|\frac{1}{3}z^{-1}\right| < 1 \qquad \left|\frac{1}{2}z^{-1}\right| < 1$$

$$|z| > \frac{1}{3} \qquad |z| > \frac{1}{2}$$

$$|z| > \mathrm{Max}\left(1/3, 1/2\right)$$



Properties of Z-transform

1. Linearity
2. Time shifting
3. Scaling in the Z-domain
4. Time reversal
5. Time expansion
6. Conjugation
7. Convolution
8. Differentiation in the Z-domain
9. Initial value theorem
10. Final value theorem

1. Linearity:

If $x(n) \xleftrightarrow{Z} X(Z)$ with ROC $= R_1$

and $y(n) \xleftrightarrow{Z} Y(Z)$ with ROC $= R_2$

then $ax(n) + by(n) \xleftrightarrow{Z} aX(Z) + bY(Z)$

with ROC atleast $R_1 \cap R_2$

Proof: We know that

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n}$$

$$\therefore \quad Z\{ax(n)+by(n)\} = \sum_{n=-\infty}^{\infty} \{ax(n)+by(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a\,x(n) z^{-n} + \sum_{n=-\infty}^{\infty} b\,y(n) z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x(n) z^{-n} + b \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$Z\{ax(n)+by(n)\} = a\,X(z) + b\,Y(z)$$

2) Time shifting:

If $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R

then $x(n-n_0) \xleftrightarrow{Z} z^{-n_0} X(z)$ with ROC = R

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

put $n - n_0 = l$

$$Z\{x(n-n_0)\} = \sum_{l=-\infty}^{\infty} x(l) \cdot z^{-(l+n_0)}$$

$$= z^{-n_0} \sum_{l=-\infty}^{\infty} x(l) z^{-l}$$

$$Z\{x(n-n_0)\} = z^{-n_0} X(z)$$

3) Scaling in the z-domain

If $x(n) \xleftrightarrow{Z} X(z)$ with ROC $= R$

then $\alpha^n x(n) \xleftrightarrow{Z} X\left(\frac{z}{\alpha}\right)$ with ROC $= |\alpha| R$

where α is a complex number

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{\alpha^n x(n)\} = \sum_{n=-\infty}^{\infty} \alpha^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (\alpha^{-1} z)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{\alpha}\right)^{-n}$$

$$Z\{\alpha^n x(n)\} = X\left(\frac{z}{\alpha}\right)$$

4) Time Reversal

If $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R

then $x(-n) \xleftrightarrow{Z} X\left(\frac{1}{z}\right)$ with ROC = $\frac{1}{R}$

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

put $l = -n$

$$Z\{x(-n)\} = \sum_{l=\infty}^{-\infty} x(l) z^{l}$$

$$Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l}$$

$$Z\{x(-n)\} = X(z^{-1}) = X\left(\frac{1}{z}\right)$$

5) Convolution

If $\quad x(n) \xleftrightarrow{Z} X(Z)$ with ROC $= R_1$

and $\quad y(n) \xleftrightarrow{Z} Y(Z)$ with ROC $= R_2$

then $x(n) * y(n) \xleftrightarrow{Z} X(Z) \cdot Y(Z)$ with ROC atleast $R_1 \cap R_2$

Proof: We know that

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

$$Z\{x(n) * y(n)\} = \sum_{n=-\infty}^{\infty} \left(x(n) * y(n)\right) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, y(n-k)\right] Z^{-n}$$

Interchanging the order of the summations

$$= \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{n=-\infty}^{\infty} y(n-k)\, z^{-n} \right]$$

put $n-k=l \Leftrightarrow n=k+l$

$$= \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{l=-\infty}^{\infty} y(l)\, z^{-l} \cdot z^{-k} \right]$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{l=-\infty}^{\infty} y(l) z^{-l}$$

$$Z\{x(n) * y(n)\} = X(z) \cdot Y(z)$$

6) Differentiation in the z - domain

If $x(n) \xleftrightarrow{z} X(z)$ with ROC = R

then $n \cdot x(n) \xleftrightarrow{z} -z \frac{dX(z)}{dz}$ with ROC = R

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Differentiating both sides with respect to 'z' we get.

$$\frac{d X(z)}{dz} = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \, (-n) z^{-n-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} [n \cdot x(n)] z^{-n}$$

$$= -z^{-1} \cdot Z\{n \cdot x(n)\}$$

$$\therefore \quad Z\{n \cdot x(n)\} = -z \frac{d}{dz} X(z)$$

7) Initial value theorem

If $x(n) = 0$: for $n < 0$ [i.e., $x(n)$ is causal]

then $\lim_{n \to 0} x(n) = x(0) = \lim_{z \to \infty} X(z)$

proof: $Z\{x(n)\} = X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$

$$X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + \cdots\cdots$$

Take limit $Z \to \infty$ on both the sides,

$$\underset{Z\to\infty}{Lt} X(Z) = \underset{Z\to\infty}{Lt} \left[x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^2} + \cdots\cdots \right]$$

$$\underset{Z\to\infty}{Lt} X(Z) = x(0) + 0 + 0 + \cdots$$

$$\therefore \underset{n\to 0}{Lt}\, x(n) = x(0) = \underset{Z\to\infty}{Lt} X(Z)$$

8) Final value theorem

If $x(n) \xleftrightarrow{Z} X(Z)$ and the poles of $X(Z)$ are all inside the unit circle, then the final value of $x(n)$ as $n \to \infty$ is given by.

$$\lim_{n \to \infty} x(n) = x(\infty) = \lim_{Z \to 1} \left[(Z-1) X(Z) \right]$$

Proof: $Z\{x(n)\} = X(Z) \longrightarrow (1)$

$Z\{x(n+1)\} = Z X(Z) - Z x(0) \longrightarrow (2)$

Now subtracting (1) from (2), we get.

$$Z[x(n+1)] - Z[x(n)] = zX(z) - zx(0) - X(z)$$

$$\sum_{n=0}^{\infty} x(n+1) z^{-n} - \sum_{n=0}^{\infty} x(n) z^{-n} = (z-1)X(z) - zx(0)$$

$$\left[x(1) + x(2) z^{-1} + x(3) z^{-2} + \cdots - x(0) - x(1) z^{-1} - x(2) z^{-2}\right]$$

$$= (z-1)X(z) - zx(0)$$

$$\lim_{z \to 1}\left[x(1) + \frac{x(2)}{z} + \frac{x(3)}{z^2} + \cdots - x(0) - \frac{x(1)}{z} - \frac{x(2)}{z^2}\right]$$

$$= \lim_{z \to 1}\left[(z-1)X(z) - zx(0)\right]$$

$$x(\infty) = \lim_{z \to 1}(z-1)X(z)$$

9) Time expansion

If $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R

then $x_{(k)}(n) \xleftrightarrow{Z} X(z^k)$ with ROC = $R^{1/k}$

where $x_{(k)}(n) = x\left(\frac{n}{k}\right)$: if n is an integer multiple of k

$= 0$; otherwise

ie $x_{(k)}(n)$ has $(k-1)$ zeros inserted between successive values of the original signal

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Similarly,

$$X(z^*) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-kn}$$

10) If $x(n) \xleftrightarrow{z} X(z)$ with ROC $= R$

then $x^*(n) \xleftrightarrow{z} X^*(z^*)$ with ROC $= R$

If $x(n)$ is real then $X(z) = X^*(z^*)$

Thus if $X(z)$ has a pole (or zero) at $z = z_0$, it must have a pole (or zero) at the complex conjugate conjugate point $z = z_0^*$.

Summary of the properties

If $x(n) \xleftrightarrow{Z} X(Z)$

Linearity $\quad a\,x(n) + b\,y(n) \xleftrightarrow{Z} a\,X(Z) + b\,Y(Z)$

Time shifting $\quad x(n-n_0) \xleftrightarrow{Z} Z^{-n_0}\,X(n)$

Scaling in the Z-domain $\quad \alpha^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{\alpha}\right)$

Time Reversal $\quad x(-n) \xleftrightarrow{Z} X\left(\frac{1}{Z}\right)$

Time expansion $\quad x_k(n) \xleftrightarrow{Z} X(Z^k)$

$x_k(n) = x\left(\frac{n}{k}\right)$ if n is an integer multiple of k

$= 0$ otherwise

Conjugation $x^*(n) \xleftrightarrow{Z} X^*(Z^*)$

Convolution $x(n) * y(n) \xleftrightarrow{Z} X(Z) \cdot Y(Z)$

Differentiation in Z-domain $n \cdot x(n) \xleftrightarrow{Z} -Z \cdot \frac{d X(Z)}{dZ}$

Initial value theorem $x(0) = \lim_{Z \to \infty} X(Z)$

Final value theorem $x(\infty) = \lim_{Z \to 1} (Z-1) X(Z)$

Example: Find the z-transform of the signals using appropriate properties

(i) $x(n) = u(-n)$

Sol^n: W.K.T $u(n) \xleftrightarrow{z} \dfrac{1}{1-z^{-1}}$ $|z|>1$

using time reversal property

$x(-n) \xleftrightarrow{z} X\left(\dfrac{1}{z}\right)$

$u(-n) \xleftrightarrow{z} \dfrac{1}{1-\left(\frac{1}{z}\right)^{-1}}$

$\left|\dfrac{1}{z}\right| > 1$

$\boxed{u(-n) \xleftrightarrow{z} \dfrac{1}{1-z}}$

$|z| < 1$

(ii) $x(n) = a^n \cdot u[n]$

Sol^n:

$$u[n] \xleftrightarrow{Z} \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1 - \left(\frac{z}{a}\right)^{-1}} \qquad |z| > a$$

scaling in z-domain

$$a^n u[n] \longleftrightarrow \frac{z}{z - a}$$

(iii) $x(n) = a^{-n} u[-n]$

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}} \qquad |z| > 1$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-\left(\frac{z}{a}\right)^{-1}}$$

scaling in z-domain

$$a^n u[n] \longleftrightarrow \frac{z}{z-a} \qquad |z| > a$$

$$a^{-n} u[-n] \longleftrightarrow \frac{\frac{1}{z}}{\frac{1}{z}-a}$$

time reversal property

$$|z| < a$$

$$a^{-n} u[-n] \xleftrightarrow{z} \frac{1}{1-az}$$

iv) $x(n) = a^{n-1} u(n-1)$

$$a^n u(n) \xleftrightarrow{Z} \frac{z}{z-a} \qquad |z| > a$$

Scaling in z-domain

$$a^{n-1} u(n-1) \xleftrightarrow{Z} \left(\frac{z}{z-a}\right) z^{-1}$$

time shift property

$$a^{n-1} u(n-1) \xleftrightarrow{Z} \frac{1}{z-a} \qquad |z| > a$$

v) $x[n] = n \cdot u[n]$

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}} \qquad |z|>1$$

$$u[n] \xleftrightarrow{z} \frac{z}{z-1}$$

$$n \cdot u[n] \xleftrightarrow{z} -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

differentiate in z-domain

$$\longleftrightarrow -z\left[\frac{(z-1) \cdot 1 - z(1)}{(z-1)^2}\right]$$

$$\longleftrightarrow -z\left[\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2}\right]$$

$$n \cdot u[n] \xleftrightarrow{z} \frac{z}{(z-1)^2} \qquad |z|>1$$

(vi)

$$x(n) = 3 \cdot 2^n u[-n]$$

Sol. This can be written as

$$x(n) = 3\left(\frac{1}{2}\right)^{-n} u(-n) \qquad (2^{-1})^{-n}$$

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{z} \frac{z/(1/2)}{\left(\frac{z}{(1/2)}-1\right)} = \frac{z}{z-1/2}$$

Scaling in the z-domain $\qquad |z| > \frac{1}{2}$

$$\left(\frac{1}{2}\right)^{-n} u(-n) \xleftrightarrow{z} \frac{z^{-1}}{z^{-1}-\frac{1}{2}} = \frac{z \cdot z^{-1}}{z\, z^{-1} - \frac{1}{2} z} = \frac{1}{1-\frac{1}{2} z}$$

$$|z| < 2$$

using time reversal property

$$3\left(\frac{1}{2}\right)^{-n} u(-n) \xleftrightarrow{z} 3 \cdot \frac{1}{1-\frac{1}{2} z} \quad |z| < 2.$$

using linearity property

vii) $x[n] = n^2\left(\frac{1}{2}\right)^n u(n-3)$

Sol^n $x[n] = n^2\left(\frac{1}{2}\right)^n u(n-3)$

multiply by $\left(\frac{1}{2}\right)^{-3}\left(\frac{1}{2}\right)^3$

$x(n) = n^2\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{-3}\left(\frac{1}{2}\right)^3 u(n-3)$

$$x[n] = \frac{1}{8} n^2 \left(\frac{1}{2}\right)^{n-3} u(n-3)$$

we know that

$$\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{z} \frac{z}{\left(z-\frac{1}{2}\right)}$$ scaling in z domain

$$\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{z} \frac{z^{-3} \cdot z}{\left(z-\frac{1}{2}\right)}$$ $|z| > \frac{1}{2}$

$$\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{z} \frac{z^{-2}}{\left(z-\frac{1}{2}\right)} = \frac{1}{z^3 - \frac{1}{2} z^2}$$

using differentiation in z-domain.

$$n\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{Z} -z\frac{d}{dz}\left[\frac{1}{z^3-\frac{1}{2}z^2}\right]$$

$$\xleftrightarrow{Z} \frac{3z^3-z^2}{\left(z^3-\frac{1}{2}z^2\right)^2}$$

Again using differentiation in Z-domain property

$$n^2\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{Z} -z\frac{d}{dz}\left[\frac{3z^3-z^2}{\left(z^3-\frac{1}{2}z^2\right)^2}\right]$$

$$\xleftrightarrow{Z} \frac{\left(z^3-\frac{1}{2}z^2\right)^2\left[9z^2-2z\right]-2\left(3z^3-z^2\right)\left[\left(z^3-\frac{1}{2}z^2\right)\left(3z^2-z\right)\right]}{\left(z^3-\frac{1}{2}z^2\right)^4}$$

$$\xleftrightarrow{Z} \frac{\left(z^3-\frac{1}{2}z^2\right)\left[9z^2-2z\right]-2\left[3z^3-z^2\right)\left(3z^2-z\right)}{\left(z^3-\frac{1}{2}z^2\right)^3}$$

$$\xleftrightarrow{Z} \frac{z^4\left(9z^2-5.5z+1\right)}{\left(z^3-\frac{1}{2}z^2\right)^3}$$

using linearity property, we get

$$x(n)=\frac{1}{8}n^2\left(\frac{1}{2}\right)^{n-3} \xleftrightarrow{Z} \frac{1}{8}\frac{z^4\left(9z^2-5.5z+1\right)}{\left(z^3-\frac{1}{2}z^2\right)^3}$$

8) $x[n] = \left[\left(\frac{1}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n) \right]$

Sln From linearity property

$$x[n] = x_1[n] + x_2[n]$$

$$X(z) = X_1(z) + X_2(z)$$

$$x_1[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| > 1$$

scaling in z-domain

$$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{z/_{1/3}}{z/_{1/3} - 1} = \frac{z}{z - 1/3}$$

$|z| > |1/3|$

$x_2(n) = \left(-\frac{1}{2}\right)^n u(n)$

same as above

$|z| > \frac{1}{2}$

$$\left(-\frac{1}{2}\right)^n u(n) \xleftrightarrow{z} \frac{z}{z+\frac{1}{2}}$$

$$\therefore \quad X(z) = X_1(z) + X_2(z)$$

$$= \frac{z}{z-\frac{1}{3}} + \frac{z}{z+\frac{1}{2}}$$

$$= \frac{z\left(z+\frac{1}{2}\right) z\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{3}\right)\left(z+\frac{1}{2}\right)} = \frac{6z^2+3z+6z^2-2z}{6z^2+z-1}$$

9) $x[n] = \cos \omega n \, u[n]$

Sol $x[n] = \left[\dfrac{e^{j\omega n} + e^{-j\omega n}}{2}\right] u[n]$

$$= \frac{1}{2}\left[e^{j\omega n} + e^{-j\omega n}\right] u[n]$$

$$= \frac{1}{2}\left[e^{j\omega n} u[n] + e^{-j\omega n} u[n]\right]$$

$$x[n] = \frac{1}{2}\left[\left(e^{j\omega}\right)^n u[n] + \left(e^{-j\omega}\right)^n u[n]\right]$$

Applying linearity property

$$x(n) = x_1(n) + x_2(n).$$

$$X(z) = X_1(z) + X_2(z)$$

$$x_1(n) = \frac{1}{2}(e^{jw})^n u[n]$$

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z|>1$$

$$(e^{jw})^n u[n] \xleftrightarrow{z} \frac{z/e^{jw}}{z/e^{jw}-1} = \frac{z}{z-e^{jw}}$$

$$\frac{1}{2}(e^{jw})^n u[n] \xleftrightarrow{z} \frac{1}{2}\left(\frac{z}{z-e^{jw}}\right)$$

Similarly

$$\frac{1}{2}(e^{-jw})^n u[n] \xleftrightarrow{z} \frac{1}{2}\left(\frac{z}{z-e^{-jw}}\right)$$

$$\therefore \quad X(z) = X_1(z) + X_2(z)$$

$$= \frac{1}{2}\left(\frac{z}{z - e^{j\omega}}\right) + \frac{1}{2}\left(\frac{z}{z - e^{-j\omega}}\right)$$

$$= \frac{1}{2}\left[\frac{z(z - e^{-j\omega}) + z(z - e^{j\omega})}{(z - e^{j\omega})(z - e^{-j\omega})}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{z^2 - z e^{-j\omega} - z e^{j\omega} + e^{j\omega} e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{-j\omega} + e^{j\omega})}{z^2 - z(e^{-j\omega} + e^{j\omega}) + 1}\right]$$

$$X(z) = \frac{1}{2}\left[\frac{2z^2 - 2z\cos\omega}{z^2 - 2z\cos\omega + 1}\right] = \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$$

10) $x[n] = a^n \cos \omega n \; u[n]$

Solution: $\cos \omega n \xleftrightarrow{z} \dfrac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$

using scaling in z-domain

$$(a)^n \cos\omega n \xleftrightarrow{z} \frac{(z/a)^2 - (z/a)\cos\omega}{(z/a)^2 - 2(z/a)\cos\omega + 1}$$

$$(a)^n \cos\omega n \xleftrightarrow{z} \frac{z^2 - az\cos\omega}{z^2 - 2az\cos\omega + a^2}$$

$|z| > a$

10) Show that for a LTI system

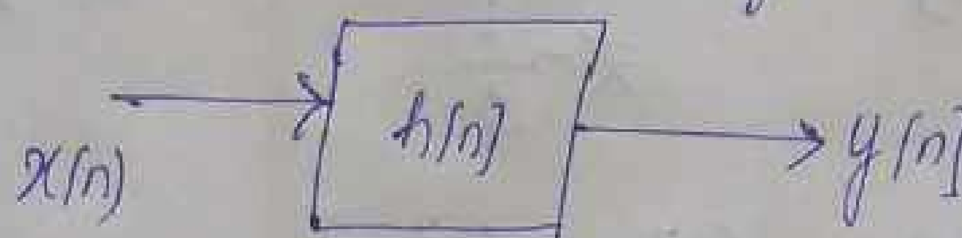
$$H(z) = \frac{Y(z)}{X(z)}$$ where $H(z) = Z\{h(n)\}$

$Y(z) = Z\{y(n)\}$

$X(z) = Z\{x(n)\}$

Solution:

Consider the LTI system



$$y[n] = x[n] * h[n]$$

Applying Z-transform on both sides

$$Z\{y(n)\} = Z\{x(n) * h(n)\}$$

using convolution property

$$Y(Z) = X(Z)\,H(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

11) Find the Z-transform of

$$x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)$$

Sol^n

using convolution property

$$Z\{x(n)\} = Z\left\{\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right\}$$

$$X(z) = Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\} \cdot Z\left\{\left(\frac{1}{3}\right)^n u(n)\right\}$$

$$Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\} = \frac{z}{z-\frac{1}{2}}$$

$$Z\left\{\left(\frac{1}{3}\right)^n u(n)\right\} = \frac{z}{z-\frac{1}{3}}$$

scaling scaling in z-domain

$$X(z) = \left(\frac{z}{z-\frac{1}{2}}\right)\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$= \frac{z^2}{z^2 - \frac{1}{2}z - \frac{1}{3}z + \frac{1}{6}}$$

$$X(z) = \frac{6z^2}{6z^2 - 5z + 1}.$$

12) If $X(z) = \dfrac{z^2 - 2z + 4}{z^2 + 3z + 7}$, find $x(0)$

Solⁿ

$$X(z) = \frac{z^2 - 2z + 4}{z^2 + 3z + 7}$$

divide by z^2

$$X(z) = \frac{1 - \frac{2}{z} + \frac{4}{z^2}}{1 + \frac{3}{z} + \frac{7}{z^2}}$$

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left[\frac{1 - \frac{2}{z} + \frac{4}{z^2}}{1 + \frac{3}{z} + \frac{7}{z^2}} \right]$$

$$\boxed{x(0) = 1.}$$

Q) Given $x(n) \xleftrightarrow{Z} X(z) = \frac{z}{z^2+4}$ with ROC : $|z| < 2$

using the z-transform properties, determine the z-transform of the following signals.

(i) $y_1(n) = 2^n x(n)$ (ii) $y_2(n) = n \cdot x(n)$

Sol: (i) $y_1(n) = 2^n x(n)$

Given $x(n) \xleftrightarrow{Z} X(z) = \frac{z}{z^2+4}$ ROC : $|z| < 2$

using scaling in the z-domain property.

$$y_1(n) = 2^n x(n) \xleftrightarrow{Z} Y_1(z) = X\left(\frac{z}{2}\right) = \frac{z/2}{\left(\frac{z}{2}\right)^2 + 4} \quad |z| < 4$$

$$= \frac{2z}{z^2+16}$$

Some standard z-transform pairs

$\delta(n)$	1	All z
$u(n)$	$\frac{z}{z-1}$	$\lvert z\rvert \geq 1$
$-u(-n-1)$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta(n-k)$	z^{-k}	All z except $z=0$
$\delta(n+k)$	z^{k}	All z except $z=\infty$
$\alpha^n u(n)$	$\frac{z}{z-\alpha}$	$\lvert z\rvert > \alpha$
$-\alpha^n u(-n-1)$	$\frac{z}{z-\alpha}$	$\lvert z\rvert < \alpha$

$n \cdot \alpha^n u(n)$	$\dfrac{\alpha z}{(z-\alpha)^2}$	$\lvert z \rvert > \alpha$
$-n\alpha^n u(-n-1)$	$\dfrac{\alpha z}{(z-\alpha)^2}$	$\lvert z \rvert < \alpha.$
$\cos wn \; u(n)$	$\dfrac{z^2 - \cos w\, z}{z^2 - 2z\cos w + 1}$	$\lvert z \rvert > 1$
$\sin wn \; u(n)$	$\dfrac{\sin w\, z}{z^2 - 2z\cos w + 1}$	$\lvert z \rvert > 1$
$\alpha^n \cos wn \; u(n)$	$\dfrac{z^2 - \alpha z\cos w}{z^2 - 2\alpha\cos w\, z + \alpha^2}$	$\lvert z \rvert > \alpha$
$\alpha^n \sin wn \; u(n)$	$\dfrac{\alpha z \sin w}{z^2 - 2\alpha\cos w\, z + \alpha^2}$	$\lvert z \rvert > \alpha$

Exp. No.

Exp. TITLE : Inverse Z-transform

To recover $x(n)$ from its Z-transform $X(Z)$, inverse Z-transform method.

We can obtain $x(n)$ from its Z-transform $X(Z)$ by using the equation.

$$x(n) = \frac{1}{2\pi j} \oint X(Z)\, Z^{n-1}\, dZ.$$

[cauchy integration].

Two alternative methods to carry out inverse Z-transform are.

→ partial fractional Expansion method.

→ Power series expansion method.

Partial fraction Expansion method.

Consider $X(z) = \frac{B(z)}{A(z)}$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$a_0 = 1$

If $M < N$ we can use partial fractions expansion directly by factorizing the den polynomial

If $M \geq N$, then by long division method bring $X(Z)$ to the form

$$X(Z) = \sum_{K=0}^{M-N} C_k Z^{-k} + \frac{\tilde{B}(Z)}{A(Z)}$$

where the numerator polynomial $\tilde{B}(Z)$ has order one less than that of denominator polynomial $A(Z)$. Then apply partial fraction for the second term in the above equation

Q) find $x(n)$ if $X(Z) = \dfrac{2Z^2}{(Z-1/2)(Z-2)}$ for the following cases of ROC

(i) $|Z| < 1/2$, (ii) $|Z| > 2$ (iii) $1/2 < |Z| < 2$

Sol $X(Z) = \dfrac{2Z^2}{(Z-1/2)(Z-2)}$

$$\frac{X(Z)}{Z} = \frac{2Z}{(Z-1/2)(Z-2)} = \frac{A}{(Z-1/2)} + \frac{B}{(Z-2)}$$

$2Z = A(Z-2) + B(Z-1/2)$

put $Z=2$

$B(2-1/2) = 4$

$B\frac{3}{2} = 4$ $B = 8/3$

put $z = 1/2$

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$$A\left(\tfrac{1}{2}-2\right) = z\left(\tfrac{1}{2}\right)$$

$$A\left(-\tfrac{3}{2}\right) = 1 \qquad \boxed{A = -\tfrac{2}{3}}$$

$$\frac{X(z)}{z} = -\frac{2}{3} \times \frac{1}{(z-1/2)} + \frac{8}{3}\,\frac{1}{(z-2)}$$

$$X(z) = -\frac{2}{3} \cdot \frac{z}{(z-1/2)} + \frac{8}{3}\,\frac{z}{z-2}$$

$$z^{-1}\left\{\frac{z}{z-1/2}\right\} = \left(\tfrac{1}{2}\right)^n u(n) \qquad |z| > |1/2|$$

$$= -\left(\tfrac{1}{2}\right)^n u(-n-1) \qquad |z| < 1/2$$

$$z^{-1}\left\{\frac{z}{z-2}\right\} = (2)^n u(n) \qquad |z| > 2$$

$$= -(2)^n u(-n-1) \qquad |z| < 2$$

i) $|Z| < 1/2$

$$Z^{-1}\{X(Z)\} = -(-2/3)(1/2)^n u(-n-1) - 8/3 (2)^n u(-n-1)$$

ii) $|Z| > 2$

$$x(n) = -2/3 (1/2)^n u(n) + 8/3 (2)^n u(n)$$

iii) $1/2 < |Z| < 2$

ARUN'S

$$x(n) = -2/3 (1/2)^n u(n) - 8/3 (2)^n u(-n-1)$$

Q)

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \qquad 0.5 < |z| < 1$$

Multiply and divide by z^2.

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

$$\frac{X(z)}{z} = \frac{z}{z^2 - 1.5z + 0.5}$$

$$z^2 - 1.5z + 0.5 = 0$$

$$z^2 - 1z - 0.5z + 0.5 = 0$$

$$z(z-1) - 0.5(z-1) = 0$$

$$(z-1)(z-0.5) = 0.$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A}{z-1} + \frac{B}{z-0.5}$$

$$A(z-0.5) + B(z-1) = z$$

put $z = 0.5$

$$B(-0.5) = 0.5 \qquad \boxed{B = -1}$$

put $z = 1$

$$A(1-0.5) = 1$$

$A\,(0.5) = 1$

$$\boxed{A = 2}$$

$$\frac{X(z)}{z} = \frac{2}{(z-1)} - \frac{1}{(z-0.5)}$$

$$X(z) = 2 \cdot \frac{z}{z-1} - \frac{z}{z-0.5}$$

$$z^{-1}\left\{\frac{z}{z-1}\right\} \quad u(n) \quad |z| > 1$$

$$-u(-n-1) \quad |z| < 1$$

$$z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n) \quad |z| > 0.5$$

$$-(0.5)^n u(-n-1) \quad |z| < 0.5$$

for $0.5 < |z| < 1$

$$x(n) = -2u(-n-1) - (0.5)^n u(n)$$

Q) $$X(z) = \frac{z(z^2 - 4z + 5)}{(z-3)(z-2)(z-1)}$$

find $x(n)$ for the following ROC's using partial fraction method

ii) $2 < |z| < 3$ iii) $|z| > 3$ iv) $|z| < 1$

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TITLE :

$$\frac{X(z)}{z} = \frac{z^2 - 4z + 5}{(z-3)(z-2)(z-1)}$$

$$\frac{X(z)}{z} = \frac{A(z-2)(z-1) + B(z-3)(z-1) + C(z-3)(z-2)}{(z-3)(z-2)(z-1)}$$

$A(z-2)(z-1) + B(z-3)(z-1) + C(z-3)(z-2)$

put $z = 2$

$4 - 8 + 5 = B(-1)(1)$

$1 = B(-1)$ $\boxed{B = -1}$

put $z = 1$

$1 - 4 + 5 = C(1-3)(1-2)$

$2 = C(-2)(-1)$

$\boxed{C = 1}$

put $z=3$

$$9-12+5 = A(3-2)(3-1)$$

$$2 = A(1)(2) \qquad \boxed{A=1}$$

$$\frac{X(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2} + \frac{1}{z-1}$$

$$X(z) = \frac{z}{z-3} - \frac{z}{z-2} + \frac{z}{z-1}.$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-3}\right\} = 3^n u(n) \qquad |z| > 3$$

$$-(3)^n u(-n-1) \qquad |z| < 3$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-2}\right\} = 2^n u(n) \qquad |z| > 2$$

$$= -(2)^n u(-n-1) \qquad |z| < 2$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = u(n) \qquad |z| > 1$$

$$= u(-n-1) \qquad |z| < 1$$

(i) $2 < |z| < 3$ $\quad |z| > 2 \quad |z| > 1$

$|z| < 3$

$$x(n) = -(3)^n u(-n-1) - (2)^n u(n) + u(n)$$

(ii) $|z| > 3; \quad |z| > 1 \quad |z| > 2.$

$$x(n) = (3)^n u(n) - (2)^n u(n) + u(n)$$

(iii) $|z| < 1 \quad ; \quad |z| < 3, \quad |z| < 2$

$$x(n) = -(3)^n u(-n-1) + (2)^n u(-n-1) - u(-n-1)$$

Q. find $x(n)$ if $X(z) = \dfrac{z(z+1)}{(z-1)(z-2)^2}$

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No.

TITLE :

$$\frac{X(z)}{z} = \frac{z+1}{(z-1)(z-2)^2} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

$$z+1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

$z = 1$

$$2 = A(1-2)^2 = A(1)$$

$$\boxed{A = 2}$$

put $z = 2$.

$$3 = C(2-1) = C(1) \qquad C = 3$$

$$z+1 = A(z^2+4-4z) + B(z^2-3z+z) + C(z-1)$$

$$z+1 = Az^2 + 4A - 4zA + Bz^2 - B3z + Bz + Cz - C$$

Equating co-efficients of z^2

$$A + B = 0$$

$$2 + B = 0$$

$$\boxed{B = -2}$$

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{3}{(z-2)^2}$$

$$X(z) = \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{3z}{(z-2)^2}$$

Assuming system to be causal,

taking inverse transform

$$x(n) = 2\,u(n) - 2(2)^n u(n) + \frac{3}{2}\cdot n\,(2)^n u(n)$$

Power series expansion

- Here X(Z) is expressed as a power series in Z^{-1} or Z. The value of the signal x(n) is then given by the co-efficients associated with Z^{-n}.

The advantage of the power series is the ability to find IZT for signals that are not a ratio of polynomials in z.

we have $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$= x(-\infty) z^{\infty} + \dots + x(-3) z^{3} + x(-2) z^{2} + x(-1) z$

$+ x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \dots + x(\infty) z^{-\infty}$ $\rightarrow (1)$

By comparing the given Z-transform with above Equation we can obtain $x(n)$ because the co-efficients of z^{-n} is the sequence values of $x(n)$, the co-efficient of z^{m} is the m^{th} term in the sequence.

This inversion method is limited to signals that are one sided i.e., ROCs of the form $|Z|<a$ or $|Z|>a$.

If $|Z|>a$, then X(Z) is expressed as a power series in Z^{-1}. If $|Z|<a$, then X(Z) is expressed as a power series in Z

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}} \qquad |z|<1$$

ROC is outside the circle, so $x(n)$ should be a right handed sequence, so convert $X(z)$ into power series having only negative powers of z as below

$$
\begin{array}{r|l}
 & 1+1.5z^{-1}+1.75z^{-2}+1.875z^{-3}+1.9375z^{-4} \\
\hline
1-1.5z^{-1}+0.5z^{-2} & 1 \\
 & 1-1.5z^{-1}+0.5z^{-2} \\
 & 1.5z^{-1}-0.5z^{-2} \\
 & 1.5z^{-1}-2.25z^{-2}+0.75z^{-3} \\
\hline
 & 1.75z^{-2}-0.75z^{-3} \\
 & 1.75z^{-2}-2.625z^{-3}+0.875z^{-4} \\
\hline
 & 1.875z^{-3}-0.825z^{-4}
\end{array}
$$

$\therefore X(z) = 1 + 1.5\,z^{-1} + 1.75\,z^{-2} + 1.875\,z^{-3} + 1.9375\,z^{-4} + \ldots$

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Comparing this with equation (1)

$x(n) = 0$; for $n < 0$

$x(0) = 1$

$x(1) = 1.5$

$x(2) = 1.75$

$x(3) = 1.875$

$x(4) = 1.9375$

$$x(n) = \{\underset{\uparrow}{1}, 1.5, 1.75, 1.875, 1.9375, \ldots\}$$

Q) Same problem with ROC: $|z| < 0.5$

Sol: Since the ROC is inside the circle of radius 0.5 in the z-plane, its corresponding $x(n)$ must be left sided sequence. Therefore convert $X(z)$ into power series having only positive powers of z as below.

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots \\
0.5z^{-2} - 1.5z^{-1} + 1 & 1 \\
 & \underline{1 - 3z + 2z^2} \\
 & 3z - 2z^2 \\
 & \underline{3z - 9z^2 + 6z^3} \\
 & 7z^2 - 6z^3 \\
 & \underline{7z^2 - 21z^3 + 14z^4} \\
 & 15z^3 - 14z^4 \\
 & \underline{15z^3 - 45z^4 + 30z^5} \\
 & 31z^4 - 30z^5
\end{array}
$$

$\therefore \quad X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots$

Comparing ~~with~~ Eqn (1)

$$x(n) = 0 \text{ for } n > 0$$

$x(0) = 0$

$x(-1) = 0$

$x(-2) = 2$

$x(-3) = 6$

$x(-4) = 14$

$x(-5) = 30$

$x(-6) = 62$

$\vdots$

$$x(n) = \{ \ldots\ldots 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0} \}$$

Q) find the inverse Z-transform of

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

$$= \left(z^2 - \tfrac{1}{2}z\right)\left(1 - z^{-2}\right)$$

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}$$

$$x(n) = \begin{cases} 1 & n = -2 \\ -1/2 & n = -1 \\ -1 & n = 0 \\ 1/2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Q. Find the inverse Z-transform of

$$X(z) = \frac{2+z^{-1}}{1-\frac{1}{2}z^{-1}} \quad \text{with ROC } |z| > \frac{1}{2}$$

using power series expansion

Sol: Since ROC indicates right handed sequence, X(z) shld be expressed as a power series in z^{-1}. Long division method is used.

$$\begin{array}{r l}
 & 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} \ldots \\
1 - \frac{1}{2}z^{-1} \,\big) & 2 + z^{-1} \\
 & -\cancel{2} \; \overset{(+)}{-} z^{-1} \\
\hline
 & 2z^{-1} \\
 & -2z^{-1} \; \overset{(+)}{-} z^{-2} \\
\hline
 & z^{-2} \\
 & -z^{-2} \; \overset{(+)}{-} \frac{1}{2}z^{-3} \\
\hline
 & \frac{1}{2}z^{-3} \\
 & (-)\,\frac{1}{2}z^{-3} \; \overset{(+)}{-} \frac{1}{4}z^{-4} \\
\hline
 & \frac{1}{4}z^{-4}
\end{array}$$

that is

$$X(z) = 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \cdots$$

$x[n] = 0 \quad n < 0.$

$x[0] = 2 \qquad x[3] = \frac{1}{2} \cdots\cdots$

$x[1] = 2$

$x[2] = 1$

(ii) ROC $< 1/2$

then the signal is left handed sequence so $X(z)$ should be a power series in z.

$$\require{cancel}
\begin{array}{r|l}
 & -2 - 8z - 16z^2 - 32z^3 - \cdots \\
-\frac{1}{2}z^{-1} + 1 & z^{-1} + 2 \\
 & \cancel{-z^{-1}} \underset{(+)}{-} 2 \\
\hline
 & \cancel{4} \\
 & \cancel{-4} \underset{(+)}{-} 8z \\
\hline
 & \cancel{8z} \\
 & \cancel{-8z} \underset{(+)}{-} 16z^2 \\
\hline
 & \cancel{16z^2} \\
 & (-)\cancel{16z^2} \underset{(+)}{-} 32z^3 \\
\hline
 & 32z^3
\end{array}$$

that is

$$X(z) = -2 - 8z - 16z^2 - 32z^3 - \cdots$$

$$x(n) = 0 \qquad n > 0.$$

$x[0] = -2$

$x[-1] = -8$

$x[-2] = -16$

$x[-3] = -32$

$\vdots$

a) $X(z) = \sum_{k=5}^{10} (1/k) z^{-k}$; $|z| > 0$

$$x(z) = \frac{1}{5} z^{-5} + \frac{1}{6} z^{-6} + \frac{1}{7} z^{-7} + \frac{1}{8} z^{-8} + \frac{1}{9} z^{-9} + \frac{1}{10} z^{-10}.$$

$$x(n) = \frac{1}{5} \delta(n-5) + \frac{1}{6} \delta(n-6) + \frac{1}{7} \delta(n-7) + \frac{1}{8} \delta(n-8) + \frac{1}{9} \delta(n-9) + \frac{1}{10} \delta(n-10)$$

$$\boxed{x(n) = \sum_{k=5}^{10} (1/k) \delta(n-k)}$$

Q) using power series expansion method, determine the z-transform of $X(z) = \cos(2z)$ $|z| < \infty$

Sol Given $X(z) = \cos(2z)$

$x(n)$ must be left sided sequence, because ROC is $|z| < \infty$, we know that

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\cos\theta = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}$$

$$\therefore X(z) = \cos 2z$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(2z)^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} \cdot z^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} (-4)^k \frac{z^{2k}}{(2k)!}$$

Taking inverse z-transform we get

$$x(n) = \sum_{k=0}^{\infty} \frac{(-4)^k}{(2k)!} \delta(n+2k)$$

Q) Determine the inverse Z-transform of

$$X(z) = \frac{1}{1-z^{-3}}$$

Sol: We know that

$$\sum_{k=0}^{\infty} \alpha^{k} = \frac{1}{1-\alpha}.$$

Given $X(z) = \dfrac{1}{1-z^{-3}} = \sum_{k=0}^{\infty} (z^{-3})^{k}$

taking inverse z - transform, we get

$$x(n) = \sum_{k=0}^{\infty} \delta(n-3k)$$

$\therefore$ $x(n) = 1$; when 'n' is integer multiple of 3 and $n \geq 0$

$= 0$; elsewhere.

Q) $X(z) = \cos(z^{-2})$; $|z| > 0$

sequence $x(n)$ must be right handed sequence because ROC is $|z| > 0$, we know that

$$\cos\theta = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}$$

$$\therefore X(z) = \cos(z^{-2}) = \sum_{k=0}^{\infty} \frac{(-1)^k (z^{-2})^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{-4k}}{(2k)!}$$

Taking inverse z-transform on b.s.

$$x(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \delta(n-4k)$$

a)

$X(z) = \ln(1+z^{-1}) \quad ; |z| > 0.$

$x(n)$ must be right handed sequence because ROC is $|z| > 0$,

we know That

$$\ln(1+\theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \cdots$$

$$\ln(1+\theta) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\theta^k}{k}.$$

$$X(z) = \ln(1+z^{-1}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z^{-1})^{k}}{k}$$

$$X(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^{-k}}{k}$$

taking IZT

$$x(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,\delta(n-k)}{k}$$

Q) Determine the inverse Z-transform of.

$$X(z) = \ln\left(\frac{\alpha}{\alpha - z^{-1}}\right) \qquad |z| > \frac{1}{|\alpha|}$$

Sol Given $X(z) = \ln\left(\frac{\alpha}{\alpha - z^{-1}}\right)$

$$= \ln\left(\frac{1}{1-(\alpha z)^{-1}}\right)$$

$$X(z) = -\ln\left(1-(\alpha z)^{-1}\right)$$

we know that

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad |x|<1.$$

$$\therefore X(z) = -\ln\left(1-(\alpha z)^{-1}\right) = \sum_{k=1}^{\infty} \frac{[(\alpha z)^{-1}]^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{\alpha^{-k}}{k} z^{-k}.$$

$(\alpha z)^{-1} < 1$

Taking IZT we get.

$|z| > \frac{1}{|\alpha|}$

$$x(n) = \sum_{k=1}^{\infty} \frac{\alpha^{-k}}{k} \delta(n-k).$$

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Transform Analysis of LTI systems

Z-transform plays an imp role in the analysis and representation of discrete time LTI systems.

Consider a discrete time LTI s/m having impulse response h(n) as shown.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = x(n) * h(n)$$

Taking Z-transform on both sides

$$Y(Z) = H(Z)\,X(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

H(Z) is referred as system function or transfer function of the system and this equation is valid for all values of Z for which X(Z) is non zero.

Q) A causal system has input $x(n)$ and output $y(n)$. find the impulse response of the s/m if

$$x(n) = \delta(n) + \frac{1}{4}\delta(n-1) - \frac{1}{8}\delta(n-2)$$

$$y(n) = \delta(n) - \frac{3}{4}\delta(n-1)$$

Sh

$$X(z) = 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}$$

$$Y(z) = 1 - \frac{3}{4}z^{-1}$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 - \frac{3}{4} z^{-1}}{1 + \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}}$$

$$= \frac{z^2 - \frac{3}{4} z}{z^2 + \frac{1}{4} z - \frac{1}{8}} = \frac{z\left(z - \frac{3}{4}\right)}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{2}\right)}$$

$$\frac{H(z)}{z} = \frac{\left(z - \frac{3}{4}\right)}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{2}\right)} = \frac{A}{\left(z - \frac{1}{4}\right)} + \frac{B}{\left(z + \frac{1}{2}\right)}$$

$$\frac{H(z)}{z} = -\frac{2}{3}\frac{1}{(z-1/4)} + \frac{5}{3}\frac{1}{(z+1/2)}$$

$$H(z) = -\frac{2}{3}\frac{z}{(z-1/4)} + \frac{5}{3}\frac{z}{(z+1/2)}$$

Since the sfm is causal

$$h(n) = -\frac{2}{3}(1/4)^n u(n) + \frac{5}{3}(-1/2)^n u(n)$$

$$\boxed{h(n) = \frac{1}{3}\left[-2(1/4)^n u(n) + 5(-1/2)^n u(n)\right]}$$

Q) We want to design a causal discrete time LTI system with the property that if the input is

$$x(n) = (1/2)^n u(n) - 1/4 (1/2)^{n-1} u(n-1),$$ then the o/p is

$$y(n) = (1/3)^n u(n).$$

Determine the impulse response h(n) and the system function H(Z) of the s/m that satisfies this cond?

Sln

Given

$$x(n) = (1/2)^n u(n) - 1/4 (1/2)^{n-1} u(n-1)$$

$$y(n) = (1/3)^n u(n)$$

$$X(Z) = \frac{Z}{Z - 1/2} - \frac{1}{4} \frac{Z \cdot Z^{-1}}{(Z - 1/2)}$$

$$= \frac{z - 1/4}{z - 1/2}.$$

$$Y(z) = \frac{z}{z - 1/3}.$$

$$H(z) = \frac{z}{(z - 1/3)} \times \frac{(z - 1/2)}{(z - 1/4)}.$$

$$\frac{H(z)}{z} = \frac{(z - 1/2)}{(z - 1/3)(z - 1/4)}$$

$$= \frac{A}{(z - 1/3)} + \frac{B}{(z - 1/4)}$$

$$\frac{H(z)}{z} = \frac{-2}{(z-1/3)} + \frac{3}{(z-1/4)}$$

$$H(z) = -2 \cdot \frac{z}{(z-1/3)} + 3 \cdot \frac{z}{(z-1/4)}$$

$$\boxed{h(n) = -2(1/3)^n u(n) + 3 \cdot (1/4)^n u(n).}$$

Q) A system has impulse response $h(n) = (1/2)^n u(n)$. Determine the input to the system if the output is given by.

$$y(n) = 1/3\, u(n) + 2/3 (-1/2)^n u(n)$$

Sln w.k. that

$$H(z) = \frac{Y(z)}{X(z)}$$

$$X(z) = \frac{Y(z)}{H(z)}$$

$$Y(z) = 1/3 \frac{z}{z-1} + 2/3 \frac{z}{z+1/2}$$

$$H(z) = \frac{z}{z + 1/2}$$

$$X(z) = \frac{\dfrac{\frac{1}{3} \cdot z}{z-1} + \dfrac{\frac{2}{3} z}{z+\frac{1}{2}}}{\dfrac{z}{z-\frac{1}{2}}}$$

$$= \frac{\dfrac{\frac{1}{3} z\left(z+\frac{1}{2}\right) + \frac{2}{3} z\,(z-1)}{(z-1)\left(z+\frac{1}{2}\right)}}{\dfrac{z}{\left(z-\frac{1}{2}\right)}}$$

$$= \frac{\dfrac{\frac{1}{3} z^2 + z\frac{1}{6} + \frac{2}{3} z^2 - \frac{2}{3} z}{(z-1)\left(z+\frac{1}{2}\right)}}{\dfrac{z}{\left(z-\frac{1}{2}\right)}}$$

$$\frac{\dfrac{z^2-\frac{1}{2}z}{(z-1)(z+\frac{1}{2})}}{\dfrac{z}{(z-\frac{1}{2})}} = \frac{z(z-\frac{1}{2})}{(z-1)(z+\frac{1}{2})} \times \frac{(z-\frac{1}{2})}{z}$$

$$\frac{Y(z)}{z} = \frac{(z-\frac{1}{2})^2}{z(z-1)(z+\frac{1}{2})}$$

$$= \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{(z+1/2)}$$

$$(z-1/2)^2 = A(z-1)(z+1/2) + B(z)(z+1/2) + C(z)(z-1)$$

$$z = 0$$

$$1/4 = A(-1)(1/2) \implies 1/4 = -A/2$$

$$\boxed{A = -1/2}$$

$$z = 1$$

$$1/4 = B(1)(3/2) \quad \boxed{B = 1/6}$$

$z = -1/2$

$1 = c(-1/2)(-1/2 - 1)$

$1 = c(-3/2)(-1/2) = c(3/4)$

$\boxed{c = 4/3}$

$$\frac{X(z)}{z} = \frac{(-1/2)}{z} + \frac{1}{6} \cdot \frac{z}{(z-1)} + \frac{4}{3} \frac{z}{(z+1/2)}$$

$$\boxed{x(n) = -1/2\, \delta(n) + 1/6\, u(n) + 4/3 \left(-1/2\right)^n u(n)}$$

Relationship between Transfer function and the difference equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$\boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}}$$

Example 1.

$$y(n) - \tfrac{1}{2}\,y(n-1) = 2x(n-1)$$

$$Y(z) - \tfrac{1}{2}\,Y(z)\,z^{-1} = 2X(z)\,z^{-1}$$

$$Y(z)\left[1 - \tfrac{1}{2}z^{-1}\right] = X(z)\,2z^{-1}$$

$$\frac{Y(z)}{X(z)} = \frac{2z^{-1}}{1 - \tfrac{1}{2}z^{-1}}$$

$$\boxed{H(z) = \frac{2}{z - \tfrac{1}{2}}}$$

$$H(z) = \frac{2 \cdot z}{z(z - 1/2)}$$

$$\frac{H(z)}{z} = \frac{A}{z} + \frac{B}{z - 1/2}$$

$$A(z - 1/2) + Bz = 2$$

$$z = 0$$

$$A(-1/2) = 2 \qquad A = -4.$$

$$z = 1/2.$$

$$B \cdot 1/2 = 2 \qquad B = -4.$$

$$H(z) = -4 \cdot \cancel{\frac{z}{z}} + 4 \cdot \frac{z}{z - 1/2}$$

$$\boxed{h(n) = -4\,\delta(n) + 4 \cdot \left(1/2\right)^n u(n)}$$

Example.2

$$y(n) - \tfrac{1}{4}\,y(n-1) - \tfrac{3}{8}\,y(n-2) = -x(n) + 2x(n-1)$$

$$Y(z) - \tfrac{1}{4}\,Y(z)\,z^{-1} - \tfrac{3}{8}\,Y(z)\,z^{-2} = -X(z) + 2X(z)\,z^{-1}$$

$$Y(z)\left[1 - \tfrac{1}{4}z^{-1} - \tfrac{3}{8}z^{-2}\right] = X(z)\left[2z^{-1} - 1\right]$$

$$\frac{Y(z)}{X(z)} = \frac{-1+2z^{-1}}{1 - \tfrac{1}{4}z^{-1} - \tfrac{3}{8}z^{-2}}$$

$$H(z) = \frac{-1 + 2z^{-1}}{1 - \tfrac{1}{4}z^{-1} - \tfrac{3}{8}z^{-2}} \times \frac{z^2}{z^2}$$

$$= \frac{-z^2 + 2z}{z^2 - \frac{1}{4}z - \frac{3}{8}} = \frac{z(-z+2)}{(z+\frac{1}{2})(z-\frac{3}{4})}$$

$$\frac{H(z)}{z} = \frac{(2-z)}{(z+\frac{1}{2})(z-\frac{3}{4})} = \frac{A}{(z+\frac{1}{2})} + \frac{B}{(z-\frac{3}{4})}$$

$$A(z-3/4) + B(z+1/2) = 2-z$$

$$z = 3/4$$

$$B(3/4 + 1/2) = 2 - 3/4$$

$$5/4 \cdot B = 5/4 \qquad \boxed{B=1}$$

$A = -2$

$$H(z) = -2 \cdot \frac{z}{(z + 1/2)} + \frac{z}{(z - 3/4)}$$

$$\boxed{h(n) = -2\left(-\frac{1}{2}\right)^n u(n) + \left(\frac{3}{4}\right)^n u(n)}$$

Stability and Causality

→ Causality and stability can be determined from the pole zero pattern and ROC of the transfe function $H(z)$.

→ for a system to be causal, $h(n) = 0$ $n<0$

ie $h(n)$ must be a right handed side sequence.
so, if the system has to be causal, then the ROC for $H(z)$ should be outside the outermost pole.

→ If the sfon has to be stable, then its impuls response h(n) must be absolutely summable. So for a causal system to be stable, the poles of $H(z)$ should lie inside the unit circle in the z-plane.

→ For a system to be both stable and causal, the ROC must include the unit circle and it must be outside the outermost pole.

(or)

→ For a sfn to be stable and causal, all the poles should lie inside the unit circle in the z-plane

Q) Determine whether the s/m described below is causal and stable.

$$H(z) = \frac{2z+1}{z^2 + z - 5/16}$$

$$H(z) = \frac{(2z+1)}{(z+5/4)(z-1/4)}$$

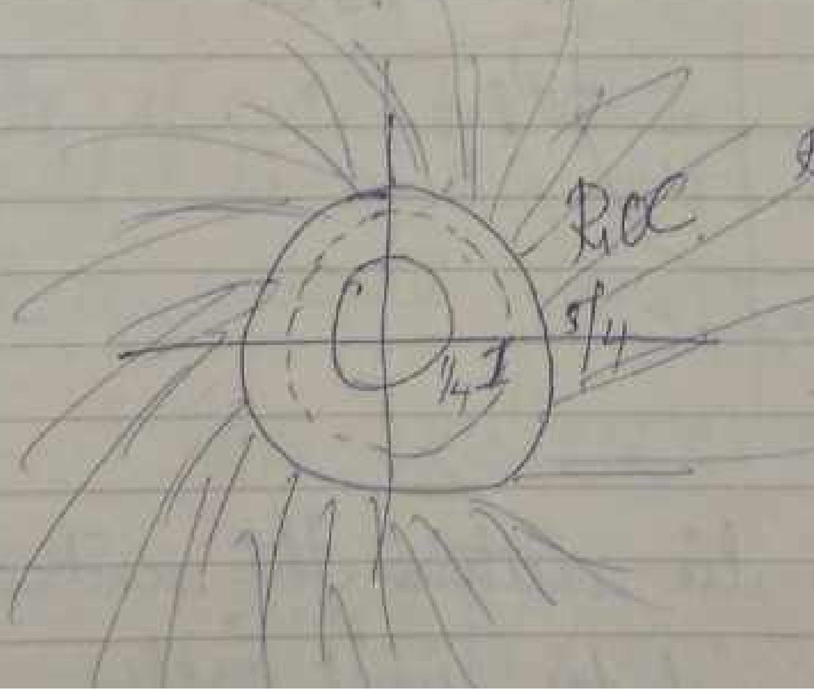
$z = -5/4$

$z = 1/4.$

$z > 5/4$

$z > 1/4$

ROC $\quad z > 5/4$



ROC is outside $5/4$, and does not include unit circle.

One of the pole lie outside the unit circle

$\therefore$ the s/m is not stable & causal.

Sol

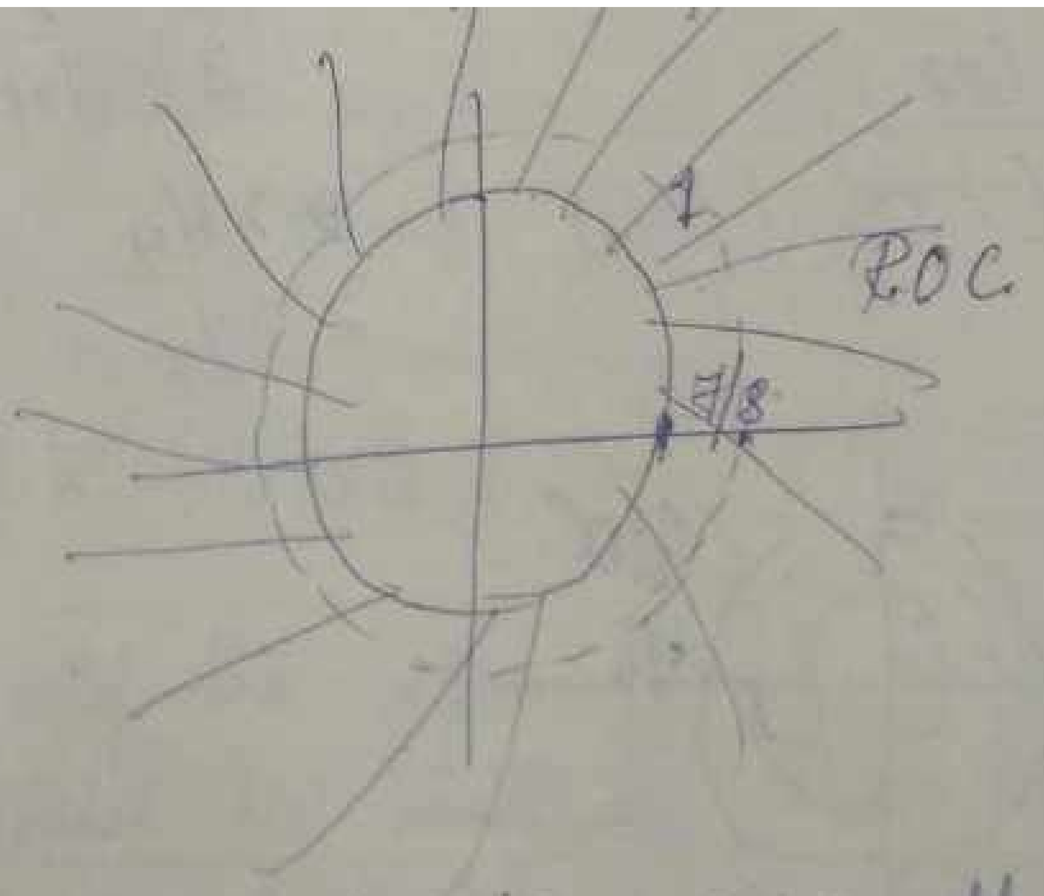
$$H(z) = \frac{1+2z^{-1}}{1+14/8\,z^{-1}+49/64\,z^{-2}} \times \frac{z^2}{z^2}$$

$$H(z) = \frac{z^2+2z}{z^2+14/8\,z+49/64}$$

$$= \frac{z(z+2)}{(z+7/8)^2}$$

$$(z+7/8)^2 = 0$$

$$z = , -7/8$$



poles lie within the unit circle and ROC also includes unit circle

$\therefore$ S/m is both stable & causal.

Q) A LTI discrete time s/m is given by the system function.

$$H(z) = \frac{3-4z^{-1}}{(1-3.5z^{-1}+1.5z^{-2})}.$$

Specify the ROC of $H(z)$ and $h(n)$ for the following conditions.

⟨i⟩ The system is stable

⟨ii⟩ The system is causal.

sln.

$$H(z) = \frac{3-4z^{-1}}{(z-1/2)(z-3)}$$

$\frac{1}{2}z^{-1} = 1$

$z^{-1} = 2$

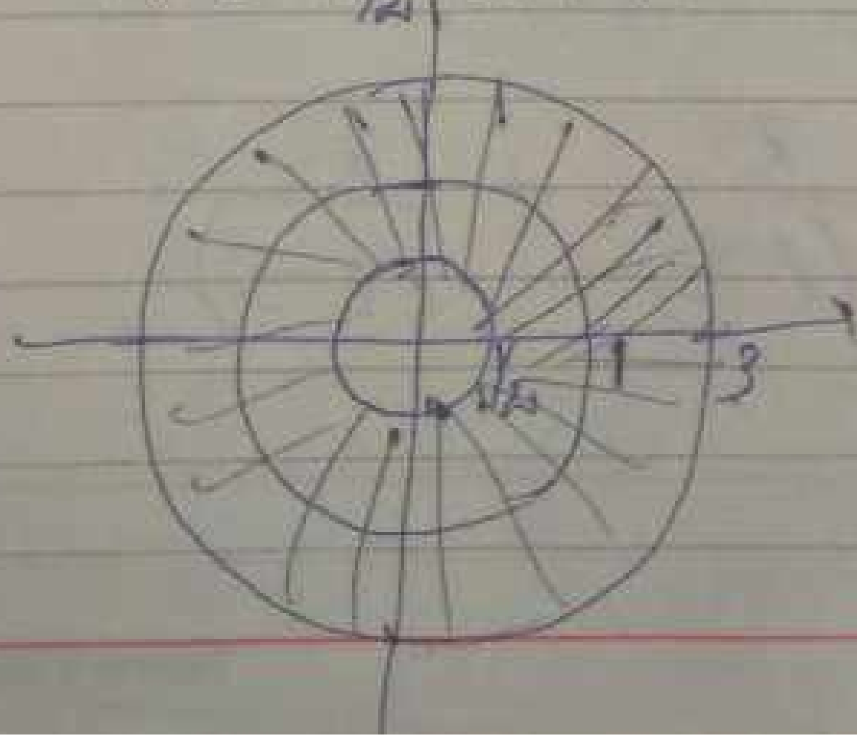
$|z| > 1/2 \qquad |z| > 3.$

$3z^{-1} = 1$

ii) for the s/m to be stable the poles should lie within the unit circle

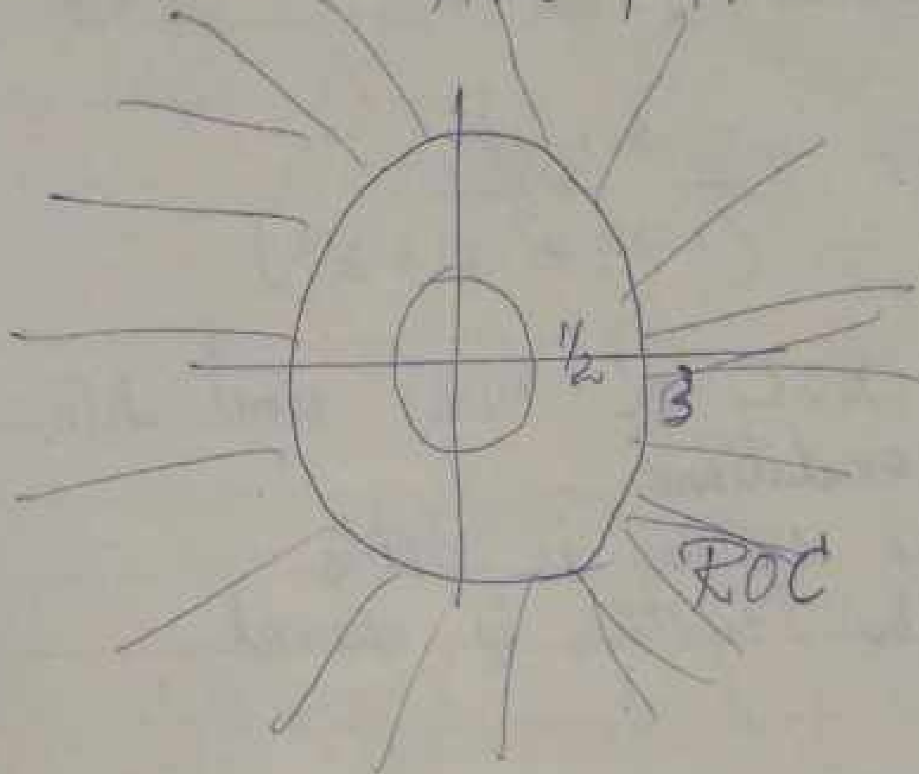
ie ROC should have unit circle.

$\therefore$ ROC : $\frac{1}{2} < |z| < 3$



ii) for the s/m to be causal, ROC should be outside the outermost pole.

$\therefore$ ROC $|z| > 3$



$$\frac{H(z)}{z} = \frac{(3z-4)}{(z-1/2)(z-3)}$$

$$= \frac{A}{(z-1/2)} + \frac{B}{(z-3)}$$

$$= \frac{1}{(z-1/2)} + \frac{2}{(z-3)}$$

$$H(z) = \frac{z}{(z-1/2)} + \frac{z}{(z-3)}.$$

i) $h(n) = (1/2)^n u(n) - 2(3)^n u(-n-1)$

ii) $= (1/2)^n u(n) + 2(3)^n u(n).$

(Q). Plot the pole zero pattern and determine which of the following systems are stable.

(i)

$$y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1)$$

$$Y(z) = Y(z)z^{-1} - 0.5Y(z)z^{-2} + X(z) + X(z)z^{-1}$$

$$Y(z)[1 - z^{-1} + 0.5z^{-2}] = X(z)[1 + z^{-1}]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

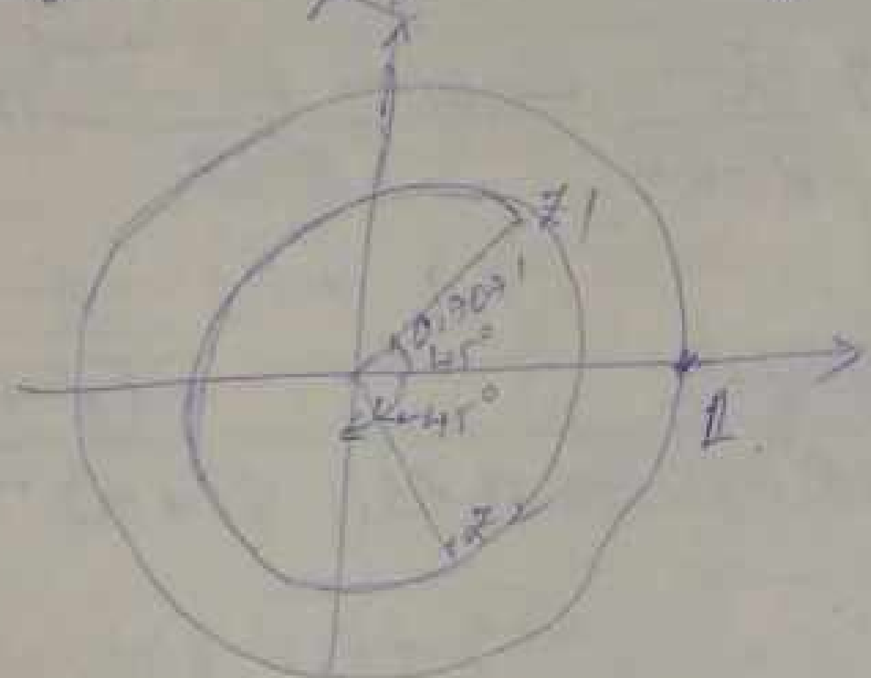
$$H(z) = \frac{z^2 + z}{z^2 - z + 0.5}$$

$$= \frac{z(z+1)}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

King

$Z_1 = 0.5 + j0.5$ $\qquad$ $Z_2 = 0.5 - j0.5$

$Z_1 = 0.7071 \angle 45^\circ$ $\qquad$ $Z_2 = 0.7071 \angle -45^\circ$



All poles are inside the unit circle,

$\therefore$ the system is stable.

2) $y(n) = 1.8\,y(n-1) - 0.72\,y(n-2) + x(n) + 0.5\,x(n-1)$

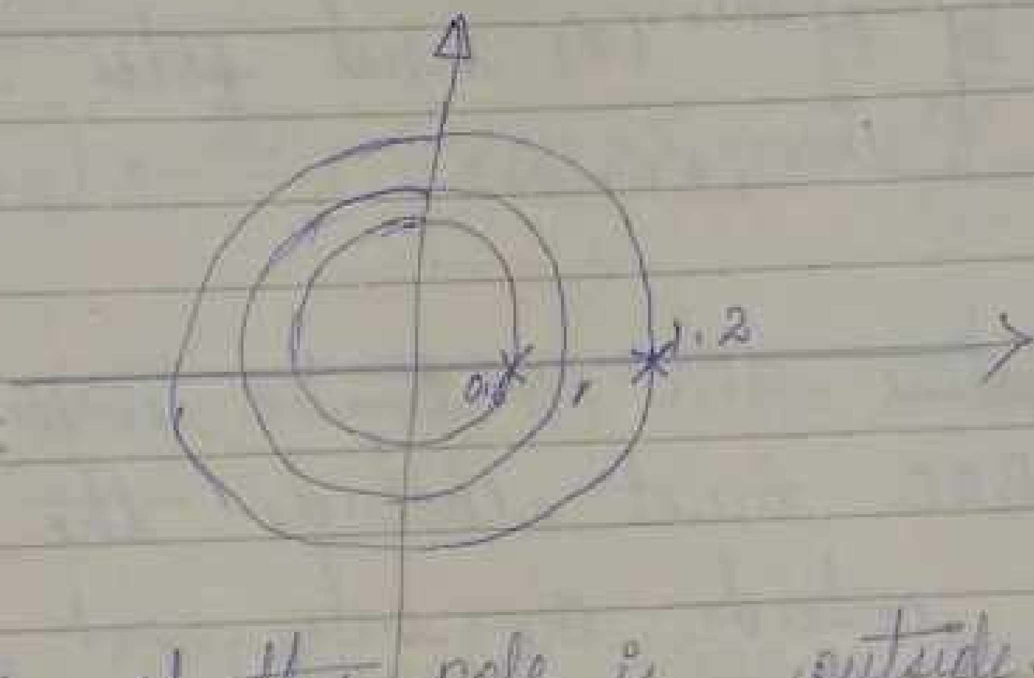
$$Y(z)\left[1 - 1.8z^{-1} + 0.72z^{-2}\right] = X(z)\left[1 + 0.5z^{-1}\right]$$

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 1.8z^{-1} + 0.72z^{-2}} \times \frac{z^2}{z^2}$$

$$H(z) = \frac{z^2 + 0.5z}{z^2 - 1.8z + 0.72}$$

$$= \frac{z(z+0.5)}{(z-1.2)(z-0.6)}$$

$Z_1 = 1.2$ $Z_2 = 0.6$



One of the pole is outside the unit circle
∴ the system is unstable.

Inverse Systems

For a system having impulse response $h(n)$, its inverse system impulse response is $h^{inv}(n)$ such that

$$h^{inv}(n) * h(n) = \delta(n).$$

Taking Z-transform on both sides

$$H^{inv}(z) \cdot H(z) = 1.$$

$$\boxed{H^{inv}(z) = \frac{1}{H(z)}}$$

∴ the transfer function of an inverse system is the inverse of the transfer function of the s[...] that we want to invert ⇒ the zeros of $H(z)$ are the poles of $H^{inv}(z)$ and poles of $H(z)$ are the zeros of $H^{inv}(z)$.

For a system $H^{inv}(z)$ to be both stable and causal its ROC must include the unit circle and it must be outside the outermost pole.

If $H^{inv}(z)$ has to be both stable and causal then all of its poles shld be inside the unit circle

Since the poles of $H^{inv}(z)$ are nothing but all the zeros of $H(z)$, for $H^{inv}(z)$ to be causal and stable, all the zeros of $H(z)$ must be inside the unit circle.

Q. for the system having transfer function

$$H(z) = \frac{1-4z^{-1}+4z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

find the transfer function of the inverse system and check whether it is both stable and causal.

Sol:

Given $$H(z) = \frac{1-4z^{-1}+4z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

The transfer function of inverse system is obtained by inverting $H(z)$

$$H^{-1}(z) = \frac{1}{H(z)} = \frac{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}{1 - 4z^{-1} + 4z^{-2}}$$

$$H^{-1}(z) = \frac{z^2 - \frac{1}{2}z + \frac{1}{4}}{z^2 - 4z + 4}.$$

$$H^{-1}(z) = \frac{z^2 - \frac{1}{2}z + \frac{1}{4}}{(z-2)^2}.$$

there are 2 poles at $z = 2$ i.e.

two poles are existing outside the unit circle.

$\therefore$ the inverse system cannot be both causal and stable.

(ii) $y(n) - \frac{1}{4}y(n-2) = 6x(n) - 7x(n-1) + 3x(n-2)$

Sln Taking Z-transform on both sides we get.

$$Y(z)\left[1 - \frac{1}{4}z^{-2}\right] = X(z)\left[6 - 7z^{-1} + 3z^{-2}\right]$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{6 - 7z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}$$

$\therefore$ the transfer function of the inverse system is

$$H^{-1}(z) = \frac{1}{H(z)} = \frac{1 - \frac{1}{4}z^{-2}}{6 - 7z^{-1} + 3z^{-3}}$$

$$H^{-1}(z) = \frac{z^2 - 1/4}{6z^2 - 7z + 3}$$

$$z = \frac{7 \pm j\sqrt{23}}{12}$$

$$|z| = 0.707$$

$\therefore$ Since both the poles exist inside the unit circle, the system can be both causal and stable.